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Some Properties and Applications of the Riemann-Hadamard Function of Darboux Problem for Telegraph Equation

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Abstract: The article deals with the problem of constructing solution of the Darboux problem for telegraph equation for the case with deviation from the characteristic. In this paper preliminarily is constructed Riemann-Hadamard function and uniqueness theorem is established for Darboux problem. Then using the function of Riemann-Hadamard was constructed a solution of the Darboux problem explicitly.

Keywords: Darboux problem, the Riemann-Hadamard function, telegraph equation

I. Introduction and Preliminary

Consider a problem of the Darboux type for the telegrath equation

$$L_0 v = v_{xx} - v_{yy} + cv = 0, \tag{1}$$

where c is an arbitrary complex number in the domain D bounded by the characteristic CB ($x + y = 1$) of equation (1), line AC ($kx - y = 0$), and by the segment AB of the axis y = 0.

Problem D **.** In the domain D find a function $v(x, y)$ which satisfies the conditions:

$$v(x, y) \in C(\overline{D}) \wedge C^1(D \cup AB) \wedge C^2(D); \tag{2}$$

$$Lv(x, y) \equiv 0, \quad (x, y) \in D; \tag{3}$$

$$v(x,0) = \tau(x), \quad 0 < x < 1; \tag{4}$$

$$v(x, kx) = \varphi(x), \quad 0 < x < \frac{1}{1+k}, \tag{5}$$

where $\tau(x)$ and $\phi(x)$ are given sufficiently smooth functions.

Definition. We call a function $v(x, y)$ quazi-regular solution of (1) if the following hold

- $v(x, y)$ satisfies (2);
- we can to applicate Green's theorem to the integrals

$$\iint_D vL_0 v dxdy, \quad \iint_D v_x L_0 v dxdy, \quad \iint_D v_y L_0 v dxdy;$$

- the boundary integrals which arise exist in the sense that: the limits taken over corresponding interior curves exist as these interior curves approach the boundary.

II. Theorem of Uniqueness

Assume v_1, v_2 : two solutions of Problem D for equation (1) and boundary conditons. Then take $v = v_1 - v_2$. In this case function $v(x, y)$ in D satisfying equation (1) and following boundary conditions

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$$\begin{cases} v = 0 \ on \ \gamma_1 \\ v = 0 \ on \ \{y=0\} \cap D \end{cases}$$

Therefore for proof uniqueness solution Problem D it is enough to show that $v = 0$ in D.

Theorem 1. *If* $v(x, y)$ − *quazi-regular solution of (1) in* D *and constant* $c > 0$, $v\big|_{\{\{y=0\}\cap D\}\cup\gamma_1} = 0$, *then* $v(x, y) \equiv 0$ *in* D.

Proof. Consider $v = v(x, y)$ be a quazi-regular of (1) defined in D. Besides consider the integral

$$2\iint_D (bv_x + av_y)(v_{xx} - v_{yy} + cv)dxdy$$

where a, b sufficiently smooth functions of (x, y).

By virtue of (1) this integral vanishes. The functions a, b are chosen in in such a way that, after a transformation of the integral by Green's formula, one obtains a positive (or non-negative) definite expression which vanishes only if $v = 0$ in D.

Consider identities

$$bv_x v_{xx} = \frac{1}{2}\left(bv_x^2\right)_x - \frac{1}{2}b_x v_x^2,$$

$$av_y v_{yy} = \frac{1}{2}\left(av_y^2\right)_y - \frac{1}{2}a_y v_y^2,$$

$$av_y v_{xx} = \left(av_x v_y\right)_x - \frac{1}{2}\left(av_x^2\right)_y + \frac{1}{2}a_y v_x^2 - a_x v_x v_y,$$

$$bv_x v_{yy} = \left(bv_x v_y\right)_y - b_y v_x v_y - \frac{1}{2}\left(bv_y^2\right)_x + \frac{1}{2}b_x v_y^2,$$

$$bv_x v = \frac{1}{2}\left(bv^2\right)_x - \frac{1}{2}b_x v^2,$$

$$av_y v = \frac{1}{2}\left(v^2\right)_y - \frac{1}{2}a_y v^2$$

Besides employ Green's theorem:

$$\oint_{\partial D} P(x, y)dx + Q(x, y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

Then employing above identities and applying Greene theorem we get:

$$J = \iint_D (b_x v_x^2 - a_y v_x^2 + 2a_x v_x v_y + b_x v_y^2 + b_x v^2 + a_y v^2)dxdy +$$

$$+ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\{y=0\}\cap D} (\ bv_x^2 + 2av_x v_y + bv_y^2 + bcv^2)dy + (av_y^2 + 2bv_x v_y + av_x^2 - acv^2)dx =$$

$$= I + J_1 + J_2 + J_3$$

$$0 = \int_{\gamma_1\cup\gamma_2\cup\{y=0\}} (bv_x^2 + 2av_x v_y + bv_y^2 + bcv^2)dy + (av_y^2 + 2bv_x v_y + av_x^2 - acv^2)dx = J_1 + J_2 + J_3$$

Finally we must choose: "nice functions" $a = a(x, y)$, $b = b(x, y)$, in D so that all conditions hold. If this occurs then uniqueness follows immediately.

Choose $a=-kb$, $b=c$. Obviously $I=0$. From $v(x,y)=0$ on $\{y=0\}\cup\gamma_1$ and the fact that

$$dy=-kdx\ (on\ \gamma_1),\ \ dy=dx\ (on\ \gamma_2)$$

we get

$$dy=-kdx \quad v=0 \quad v_y dy+v_x dx=0$$

$$J_1=\int_{\gamma_1}(kb+a)(1-k^2)(v_y^2-cv^2)dx=0$$

$$J_2=\int_{\gamma_2}[c(1-k)(v_x+v_y)^2+c^2v^2(1+k)]dx\ge 0$$

$$J_3=-kc\int_{\{y=0\}\cap D}v_y^2 dx\ge 0$$

III. Construction Function Riemann-Hadamard

On the plane (x,y) we pass to the characteristic coordinates $\xi=x+y$, $\eta=x-y$. Then equation (1) takes the form

$$Lu=u_{\xi\eta}+\frac{c}{4}u=0$$

where

$$u(\xi,\eta)=u\left(\frac{1}{2}(\xi+\eta),\frac{1}{2}(\xi-\eta)\right)$$

and the domain D is mapped to the domain

$$\Delta=\left\{(\xi,\eta)\middle|0<\xi<\eta<\alpha\xi<1,\alpha=\frac{1-k}{1+k}>1\right\}$$

and, respectively, Problem D is posed as follows:

Problem D'**.** In the domain Δ find a function $v(x,y)$ which satisfies the conditions

$$u(\xi,\eta)\in C(\overline{\Delta})\wedge C^1(\Delta\cup\{\eta=\alpha\xi\})\wedge u_{\xi\eta}\in C(\Delta);$$

$$Lu(\xi,\eta)\equiv 0,(\xi,\eta)\in\Delta,$$

$$u(\xi,\alpha\xi)=\tau(\xi),0\le\xi\le\frac{1}{\alpha};$$

$$u(\xi,\xi)=\psi(\xi),0\le\xi\le 1;$$

$$\psi(0)=\varphi(0).$$

It is well known that the Riemann-–Hadamard function plays an important role in the study of problem D'; this function was defined and constructed in [1-6] for some special cases of Eq.(1).In this section, we present an in a sense modified (as compared with the approaches used in the above-mentioned papers) approach to defining the Riemann-–Hadamard function of problem D' for Eq. (1) in case if boundary values is defined on non-characteristic.

Let domain Δ is divided into following subdomains

$$\sigma_0=\left\{(\xi,\eta)\,|\,\eta<\alpha\xi,\xi<\frac{\eta_0}{\alpha},\eta>\xi_0\right\},$$

$$\sigma_{2k}=\left\{(\xi,\eta)\,|\,\eta<\alpha\xi,\xi<\frac{\eta_0}{\alpha^{k+1}},\eta>\frac{\xi_0}{\alpha^k}\right\},$$

$$\sigma_{2k+1} = \left\{(\xi,\eta) \mid \eta < \alpha\xi, \xi < \frac{\xi_0}{\alpha^{k+1}}, \eta > \frac{\eta_0}{\alpha^{k+1}}\right\},$$

$$\omega_0 = \left\{(\xi,\eta) \mid \eta < \eta_0, \xi < \xi_0, \eta > \xi_0, \xi > \frac{\eta_0}{\alpha}\right\},$$

$$\omega_{2k} = \left\{(\xi,\eta) \mid \xi > \frac{\eta_0}{\alpha^{k+1}}, \xi < \frac{\xi_0}{\alpha^k}, \eta < \frac{\eta_0}{\alpha^k}, \eta > \frac{\xi_0}{\alpha^k}\right\},$$

$$\omega_{2k+1} = \left\{(\xi,\eta) \mid \xi < \frac{\eta_0}{\alpha^{k+1}}, \xi > \frac{\xi_0}{\alpha^{k+1}}, \eta > \frac{\eta_0}{\alpha^{k+1}}, \eta < \frac{\xi_0}{\alpha^k}\right\},$$

$$\Delta_{2k+1} = \left\{(\xi,\eta) \mid \eta > \xi, \xi > \frac{\eta_0}{\alpha^{k+1}}, \eta < \frac{\xi_0}{\alpha^k}\right\},$$

$$\Delta_{2k} = \left\{(\xi,\eta) \mid \eta > \xi, \xi > \frac{\xi_0}{\alpha^k}, \eta < \frac{\eta_0}{\alpha^k}\right\},$$

$$k = 1,2,....$$

In what follows, we assume that function is known as the Riemann–Hadamard function $R(\xi,\eta;\xi_0,\eta_0)$ satisfies conditions

$$1.\ LR(\xi,\eta;\xi_0,\eta_0) = R_{\xi\eta} + cR = 0.$$

$$2.\ R_\xi \mid_{\eta=\eta_0} = 0, \quad R_\eta \mid_{\xi=\xi_0} = 0, \quad R \mid_{\eta=\xi \cup \eta=\alpha\xi} = 0.$$

$$3.\ \frac{\partial[R_1]}{\partial\xi} = 0, [R_1] = \lim_{\varepsilon\to 0}\left[R\left(\xi; \frac{\xi_0}{\alpha^{k-1}} + \varepsilon; \xi_0; \eta_0\right) - R\left(\xi; \frac{\xi_0}{\alpha^{k-1}} - \varepsilon; \xi_0; \eta_0\right)\right]$$

$$\frac{\partial[R_2]}{\partial\xi} = 0, [R_2] = \lim_{\varepsilon\to 0}\left[R\left(\xi; \frac{\eta_0}{\alpha^k} + \varepsilon; \xi_0; \eta_0\right) - R\left(\xi; \frac{\eta_0}{\alpha^k} - \varepsilon; \xi_0; \eta_0\right)\right]$$

$$\frac{\partial[R_3]}{\partial\eta} = 0, [R_3] = \lim_{\varepsilon\to 0}\left[R\left(\frac{\xi_0}{\alpha^k} + \varepsilon; \eta; \xi_0; \eta_0\right) - R\left(\frac{\xi_0}{\alpha^k} - \varepsilon; \eta; \xi_0; \eta_0\right)\right]$$

$$\frac{\partial[R_4]}{\partial\xi} = 0, [R_4] = \lim_{\varepsilon\to 0}\left[R\left(\frac{\eta_0}{\alpha^k} + \varepsilon; \xi\xi_0; \eta_0\right) - R\left(\frac{\eta_0}{\alpha^k} - \varepsilon; \xi; \xi_0; \eta_0\right)\right]$$

$$4.\ R(\xi,\eta,\xi_0,\eta_0) = 1$$

$$k = 1,2,....$$

Then the function of the Riemann-Hadamard is determined by the recurrent formulas as follows

$$R_{\sigma_{2k}} = R_{\omega_{2k}} - J_0\left(\sqrt{c\left(\eta\alpha^{-k-1} - \xi_0\right)\left(\alpha^{k+1}\xi - \eta_0\right)}\right), (\xi,\eta) \in \sigma_{2k},$$

$$R_{\sigma_{2k+1}} = R_{\omega_{2k+1}} + J_0\left(\sqrt{c(\alpha^{k+1}\xi - \xi_0)\left(\eta\alpha^{-k-1} - \eta_0\right)}\right), (\xi,\eta) \in \sigma_{2k+1},$$

$$R_{\Delta_{2k+1}} = R_{\omega_{2k}} - J_0\left(\sqrt{c(\alpha^k\eta - \xi_0)\left(\xi\alpha^{-k} - \eta_0\right)}\right), (\xi,\eta) \in \Delta_{2k+1},$$

$$R_{\Delta_{2k+2}} = R_{\omega_{2k+1}} + J_0\left(\sqrt{c(\alpha^{k+1}\eta - \eta_0)\left(\xi\alpha^{-k-1} - \xi_0\right)}\right), (\xi,\eta) \in \Delta_{2k+2},$$

$$R_{\omega_{2k+1}} = R_{\Delta_{2k+1}} - J_0\left(\sqrt{c(\alpha^{k+1}\xi - \eta_0)(\eta\alpha^{-k-1} - \xi_0)}\right), (\xi,\eta) \in \omega_{2k+1},$$

$$R_{\omega_{2k+2}} = R_{\Delta_{2k+2}} - J_0\left(\sqrt{c(\alpha^{k+1}\xi - \xi_0)(\eta\alpha^{-k-1} - \eta_0)}\right), (\xi,\eta) \in \omega_{2k+2},$$

where $J_0(\cdot)$ is Bessel function of zero order.

IV. Construction Solution of Darboux Problem

One can readily see that

$$u \cdot LR - R \cdot Lu = \frac{1}{2}\left(uR_\eta - Ru_\eta\right)_\xi + \frac{1}{2}\left(uR_\xi - Ru_\xi\right)_\eta \tag{6}$$

By using relation (6), where u is a regular solution of Eq. (1) in the domain Δ and R is function of Riemann-Hadamard, and by applying the Green formula to the above-mentioned subdomains ω_k, σ_k, Δ_k of the domain Δ, one can readily justify the relations

$$0 = \int_{(\cup\partial\Delta_i \cup \partial\omega_k \cup \partial\sigma_m)} \left(uR_\xi - Ru_\xi\right)d\xi - \left(uR_\eta - Ru_\eta\right)d\eta =$$

$$= I_{ED} + I_{DC} + I_{CA} + I_{AE},$$

where $D = (\xi_0, \eta_0)$, $C = \left(\frac{\eta_0}{\alpha}, \eta_0\right)$, $B = \left(\frac{\xi_0}{\alpha}, \xi_0\right)$, $A = (0,0)$, $E = (\xi_0, \xi_0)$.

Calculating the integrals $I_{ED}, I_{DC}, I_{CA}, I_{AE}$, one obtains:

$$I_{ED} = uR_1\Big|_E^D = uR_1(D) - uR_1(E) = u(\xi_0, \eta_0) - \tau(\xi_0)$$

$$I_{DC} = -uR_1\Big|_D^C = -uR_1(C) + uR_1(D) = -\psi\left(\frac{\eta_0}{\alpha}\right) + u(\xi_0, \eta_0)$$

Since $d\eta = \alpha d\xi$ on AC then

$$I_{AC} = \int_0^{\frac{\eta_0}{\alpha}} u\left[(R_\xi - \alpha(R_\eta\right]d\xi =$$

$$= \sum_{n=1}^{\infty}\left(\int_0^{\frac{\eta_0}{\alpha^n}} u\left[(R_{\sigma_{2n-2}})_\xi - \alpha(R_{\sigma_{2n-2}})_\eta\right]d\xi + \int_0^{\frac{\xi_0}{\alpha^n}} u\left[(R_{\sigma_{2n-1}})_\xi - \alpha(R_{\sigma_{2n-1}})_\eta\right]d\xi\right) =$$

$$= \sum_{n=1}^{\infty} A\left(\eta_0 - \alpha^{2n-1}\xi_0\right)\int_0^{\frac{\eta_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \eta_0)(\xi\alpha^{1-n} - \xi_0)}\right)}{\sqrt{(\alpha^{n-1}\xi - \alpha^{2n-2}\xi_0)(\alpha^n\xi - \eta_0)}}\tau(\xi)d\xi +$$

$$+ \sum_{n=1}^{\infty} A\left(\alpha^{2n-1}\eta_0 - \xi_0\right)\int_0^{\frac{\xi_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \xi_0)(\xi\alpha^{1-n} - \eta_0)}\right)}{\sqrt{(\alpha^{n-1}\xi - \alpha^{2n-2}\eta_0)(\alpha^n\xi - \xi_0)}}\tau(\xi)d\xi$$

Again, on AE, $d\eta = d\xi$. Hence

$$I_{AE} = \int_0^{\xi_0} u\left(R_\xi - R_\eta\right) d\xi = \int_0^{\xi_0} u\left[(R_{\Delta_1})_\xi - (R_{\Delta_1})_\eta\right] d\xi +$$

$$+\sum_{n=1}^{\infty}\left(\int_0^{\frac{\eta_0}{\alpha^n}} u\left[(R_{\Delta_{2n}})_\xi - \alpha (R_{\Delta_{2n}})_\eta\right] d\xi + \int_0^{\frac{\xi_0}{\alpha^n}} u\left[(R_{\Delta_{2n+1}})_\xi - \alpha (R_{\Delta_{2n+1}})_\eta\right] d\xi\right) =$$

$$= A\left(\eta - \xi\right)\int_0^{\xi} \frac{J_1\left(\sqrt{c(t-\xi)(t-\eta)}\right)}{\sqrt{\left(t-\xi\right)\left(t-\eta\right)}}\psi(t)dt +$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n}\xi_0 - \eta_0\right)\int_0^{\frac{\eta_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \eta_0)\left(\xi\alpha^{-n} - \xi_0\right)}\right)}{\sqrt{\left(\alpha^n\xi - \alpha^{2n}\xi_0\right)\left(\alpha^n\xi - \eta_0\right)}}\psi(\xi)d\xi +$$

$$+\sum_{n=1}^{\infty} A\left(\xi_0 - \alpha^{2n}\eta_0\right)\int_0^{\frac{\xi_0}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n\xi - \xi_0)\left(\xi\alpha^{-n} - \eta_0\right)}\right)}{\sqrt{\left(\alpha^n\xi - \alpha^{2n}\eta_0\right)\left(\alpha^n\xi - \xi_0\right)}}\psi(\xi)d\xi$$

Therefore, one can readily show that the solution of the problem D' can be represented at the point $(\xi,\eta)\in\Delta$ in the form

$$u(\xi,\eta) = \tau(\xi) + \psi\left(\frac{\eta}{\alpha}\right) + A\left(\eta - \xi\right)\int_0^{\xi} \frac{J_1\left(\sqrt{c(t-\xi)(t-\eta)}\right)}{\sqrt{\left(t-\xi\right)\left(t-\eta\right)}}\psi(t)dt +$$

$$+\sum_{n=1}^{\infty} A\left(\eta - \alpha^{2n-1}\xi\right)\int_0^{\frac{\eta}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t - \eta)\left(t\alpha^{1-n} - \xi\right)}\right)}{\sqrt{\left(\alpha^{n-1}t - \alpha^{2n-2}\xi\right)\left(\alpha^n t - \eta\right)}}\tau(t)dt +$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n-1}\eta - \xi\right)\int_0^{\frac{\xi}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t - \xi)\left(t\alpha^{1-n} - \eta\right)}\right)}{\sqrt{\left(\alpha^{n-1}t - \alpha^{2n-2}\eta\right)\left(\alpha^n t - \xi\right)}}\tau(t)dt +$$

$$+\sum_{n=1}^{\infty} A\left(\alpha^{2n}\xi - \eta\right)\int_0^{\frac{\eta}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t - \eta)\left(t\alpha^{-n} - \xi\right)}\right)}{\sqrt{\left(\alpha^n t - \alpha^{2n}\xi\right)\left(\alpha^n t - \eta\right)}}\psi(t)dt +$$

$$+\sum_{n=1}^{\infty} A\left(\xi - \alpha^{2n}\eta\right)\int_0^{\frac{\xi}{\alpha^n}} \frac{J_1\left(\sqrt{c(\alpha^n t - \xi)\left(t\alpha^{-n} - \eta\right)}\right)}{\sqrt{\left(\alpha^n \xi - \alpha^{2n}\eta\right)\left(\alpha^n t - \xi\right)}}\psi(t)dt \tag{7}$$

Theorem 2. 1*If functions* $\tau(\xi)\in C^1\left[0,\frac{1}{\alpha}\right]$, $\alpha > 1$, $\psi(\xi)\in C^2[0,1]$ *and* $c > 0$ *there exist a uniqueness solution to the problem* D' *of the form (7).*

V. Conclusions

In this article, we discussed a little-known method of constructing the Riemann-Hadamard function for telegraph equation for the case with deviation from the characteristic. This result is a new study on the issue.

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