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Tail Conditional Expectations for Extended Exponential Dispersion Models

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Abstract: The measurement of the tail conditional expectation, also called tail value-at-risk, helps insurance companies determine the amount of capital needed to pay claims resulting from catastrophic events when premium revenues are insufficient. In this paper, we extend the exponential dispersion models and derive tail conditional expectation forms of the new models that generalize results and discussions of paper by Landsman and Valdez (2005).

Keywords: Exponential dispersion model, tail conditional expectations, tail value-at-risk

I. INTRODUCTION

Insurance companies often need to determine the amount of capital to pay claims resulting from catastrophic events when premium revenues are insufficient. Determining these amounts is a difficult process. We need to model the probability distribution of the incurred and occurring losses. More importantly, we should determine the best risk measurement [1] to find the amount of loss to cover the claims. Tail conditional expectation (TCE) has become increasingly popular for measuring this kind of risk, especially the adequacy of existing capital and the possibility of financial ruin [3, 4, 9]. It is one of the newest risk modeling techniques adopted by the insurance industry.

Suppose that for some period of time, an insurance company faces the risk of losing an amount of *X*. Its distribution function and tail function are denoted by $F_X(x) = \Pr(X \le x)$ and $S_X(x) = \Pr(X > x)$. The function $S_X(x)$ is also called the survival function in life contingency literature. The *tail conditional expectation* of *X* is defined as

$$TCE_X(x) = E(X \mid X > x), \tag{1.1}$$

Which is also called the tail value at risk (TVaR). We can interpret this risk measure as the mean of worst losses beyond a certain level. The formula used to evaluate TCE is

$$TCE_X(x) = \frac{1}{S_X(x)} \int_x^{\infty} t dF_X(t), \tag{1.2}$$

Provided that $S_X(x) > 0$.

The value of x is usually set based on the potential value of the loss x_q of the distribution with the property

$$S_X(x_q) = 1 - q,$$

where $0 < q < 1$. With a large value of q, the value x_q is considered to be the amount of loss unlikely to happen in a normal situation, but could potentially ruin the business. It is therefore important for a company to monitor and prepare for such an extreme situation. TCE provides a good measure for this purpose.

The class of exponential dispersion models has served as "error distributions" for generalized linear models in the sense developed by [13], which includes many well known distributions such as: Poisson, Binomial, Normal, Gamma etc. Also, credibility formulas of exponential dispersion models given by [8, 10, 11, 14] preserve the property of a

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predictive mean. A thorough and systematic investigation of exponential dispersion models was done by Jorgensen [5, 6, 7]. In [12], the reproductive and additive models [see (2.1), (2.2)] and their tail conditional expectations are studied. In this paper, we further extend the models to an even more general form [see (3.1)], and investigate the tail conditional expectation.

we provide a brief review of the exponential dispersion family (EDF). We study the generalized exponential dispersion family, investigate its properties, and derive the formula for its tail conditional expectation. Finally, we summarize conclusions.

II. Exponential Dispersion Models

A random variable X is said to belong to the EDF of distributions if its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q_λ and can be represented as follows.

$$dP_{\theta,\lambda} = e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x), \tag{2.1}$$

for some function κ(θ). The parameter θ is named the canonical parameter belonging to the set

$\Theta = \{\theta \in R \mid \kappa(\theta) < \infty\}$.

The parameter λ is called the index parameter belonging to the set of positive real numbers $\Lambda = \{\lambda \mid \lambda > 0\} = R_+$. The representation in (2.1) is called the *reproductive* form of EDF and is denoted by $X \sim ED(\theta,\lambda)$ for a random variable belonging to this family. With the transformation of $Y = \lambda X$, a so-called *additive* form can be obtained. Its probability measure $P^*_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q^*_λ, which can be represented as

$$dP^*_{\theta,\lambda} = e^{[\theta y - \lambda\kappa(\theta)]} dQ^*_\lambda(y), \tag{2.2}$$

If the measure Q_λ in (2.1) is absolutely continuous with respect to a Lebesgue measure, then the density of X has the form

$$f_X(x) = e^{\lambda[\theta x - \kappa(\theta)]} q_\lambda(x). \tag{2.3}$$

The same can be said about additive model, $ED^*(\theta,\lambda)$, and Y has the density

$$f_Y(y) = e^{[\theta y - \lambda\kappa(\theta)]} q^*_\lambda(y), \tag{2.4}$$

Consider the loss random variable X belonging to the family of exponential dispersion models in reproductive or additive form. We will denote the tail probability function by $S(\cdot \mid \theta,\lambda)$. This simplifies the notation by dropping the subscript when no confusion can happen and emphasizes its dependence on the parameters θ and λ.

In [12], the TCEs are given for both reproductive form and additive form of exponential dispersion models. Suppose that the random variable X belongs to EDF. If the survival function has partial derivative with respect to θ, $\kappa(\theta)$ is a differentiable function, and one can differentiate the survival function $S(\cdot \mid \theta,\lambda)$ in θ under the integral sign, then

- For $X \sim ED(\mu,\lambda)$, the reproductive form of EDF,

$$TCE_X(x) = \mu + \frac{h}{\lambda}, \tag{2.5}$$

Where

$$h = \frac{\partial}{\partial\theta} \log S(x \mid \theta,\lambda).$$

- For $X \sim ED^*(\mu,\lambda)$, the additive form of EDF,

$$TCE_X(x) = \mu + h, \tag{2.6}$$

In this paper, we will extend the formulas of exponential dispersion models and investigate their properties. The first generalization we would consider is to replace the multiplier θ in EDF by a function of θ. This is motivated by the linear exponential family (LEF) [2], which takes the form

$$f(x|\theta) = \frac{e^{r(\theta)x} q(x)}{p(\theta)} \tag{2.7}$$

With this generalization, the extended model includes LEF as a special case.

The second generalization is to replace the term $\lambda\kappa(\theta)$ by a general bivariate function of λ and θ. Note that

$$\lambda\kappa(\theta) = \log(\int_R e^{\lambda\theta x} dQ_\lambda(x)) \tag{2.8}$$

for the reproductive form of EDF and

$$\lambda\kappa(\theta) = \log(\int_R e^{\theta y} dQ_\lambda^*(y)) \tag{2.9}$$

for the additive form of EDF. For general distributions Q_λ or Q_λ^*, the functions on the right of (2.8) or (2.9) should be a general bivariate function, not necessarily able to be written in the form of $\lambda\kappa(\theta)$. Therefore in EDF, the choice of the distributions Q_λ or Q_λ^* are restricted. The generalization of the model removes the restriction and allows us to consider more general distributions.

What we want to do in this paper is to combine the two generalizations and develop an extended dispersion model, as well as study its properties and derive its TCE representation. Such a family of models will be referred to as the generalized exponential dispersion family (GEDF).

III. Generalized Exponential Dispersion Family

We consider the generalization of combining the reproductive form (2.1) and additive (2.2) as

$$dP_{\theta,\lambda}^{**} = e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} dQ_\lambda^{**}(x). \tag{3.1}$$

If the measure Q_λ is absolutely continuous with respect to the Lebesgue measure, then the density of *X* has the form

$$f_X(x) = e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} q_\lambda^{**}(x). \tag{3.2}$$

3.1. Mean and Variance of GEDF

Theorem1. *Suppose that a random variable X belongs to the GEDF whose distribution is given by* (3.1). *If its probability measure* $P_{\theta,\lambda}^{**}$ *is absolutely continuous with respect to some measure* Q_λ^{**}, $r(\theta)$ *has second derivative and is invertible,* $\kappa(\lambda,\theta)$ *has the second partial derivative to* θ, *then the mean value of X is*

$$\mu = \frac{1}{\lambda r'(\theta)} \frac{\partial \kappa(\lambda,\theta)}{\partial\theta}, \tag{3.3}$$

And the variance of X is

$$\mathrm{Var}(X) = \frac{1}{\lambda^2 r'(\theta)} [\frac{1}{r'(\theta)} \frac{\partial \kappa(\lambda,\theta)}{\partial\theta}]'. \tag{3.4}$$

Proof. The generating function can be derived as follows

$$K_X(t) = \log E(e^{Xt}) = \log\{\int_R e^{\lambda[(t/\lambda)+r(\theta)]x-\kappa(\lambda,\theta)} dQ_\lambda^{**}(x)\},$$

Let ξ be the number such that $r(\theta)+t/\lambda = r(\xi)$. Since $r(\theta)$ is invertible, we have $\xi = r^{-1}[r(\theta)+(t/\lambda)]$. Then

$$K_X(t) = \kappa(\lambda, r^{-1}[r(\theta)+(t/\lambda)] - \kappa(\lambda,\theta).$$

It follows that its moment generating function can be written as

$$M_X(t) = e^{\kappa(\lambda, r^{-1}(r(\theta)+t/\lambda))-\kappa(\lambda,\theta)}. \tag{3.5}$$

Knowing that $(f^{-1}(x))' = 1/f'(f^{-1}(x))$ for an invertible differentiable function, we can find the first order derivative of the moment generating function:

$$M'_x(t) = e^{\kappa(\lambda, r^{-1}(r(\theta)+t/\lambda))-\kappa(\lambda,\theta)} \frac{\partial\kappa(\lambda, r^{-1}(r(\theta)+t/\lambda))}{\partial\theta} \frac{1}{\lambda r'(r^{-1}(r(\theta)+\mathrm{t}/\lambda))}$$

When t = 0, we obtain

$$M'_x(0) = e^{\kappa(\lambda, r^{-1}(r(\theta)))-\kappa(\lambda,\theta)} \frac{\partial\kappa(\lambda, r^{-1}(r(\theta)))}{\partial\theta} \frac{1}{\lambda r'(r^{-1}(r(\theta)))},$$

which gives the mean value

$$\mu = \frac{1}{\lambda r'(\theta)} \frac{\partial\kappa(\lambda,\theta)}{\partial\theta}.$$

For the variance of the combination function, we have

$$E(X^2) = M''_x(0) = \frac{1}{\lambda^2}\{[\frac{1}{r'(\theta)}\frac{\partial\kappa(\lambda,\theta)}{\partial\theta}]^2 + \frac{1}{[r'(\theta)]^2}\frac{\partial^2\kappa(\lambda,\theta)}{\partial\theta^2} - \frac{r''(\theta)}{[r'(\theta)]^3}\frac{\partial\kappa(\lambda,\theta)}{\partial\theta}\}.$$

Then, we can get the variance as

$$\mathrm{Var}(X) = \frac{1}{\lambda^2 r'(\theta)}[\frac{1}{r'(\theta)}\frac{\partial\kappa(\lambda,\theta)}{\partial\theta}]'.$$

This completes the proof of the theorem.

When $\lambda = 1$ and $\kappa(\lambda,\theta) = \kappa(\theta) = \log(p(\theta))$, the GEDF reduces to the LEF (2.7). Theorem 1 gives

$$\mu = \mu(\theta) = \frac{\kappa'(\theta)}{r'(\theta)} = \frac{p'(\theta)}{r'(\theta)p(\theta)}$$

and $\mathrm{Var}(X) = \dfrac{\mu'(\theta)}{r'(\theta)}$. These coincide with the results for the linear exponential family in [9].

3.2. TCE of GEDF

Theorem2. *Under the same assumptions as in Theorem 1, if one can differentiate the tail function* $S(\cdot|\theta,\lambda)$ *in* θ *under the integral sign, then the tail conditional expectation is given by period*

$$TCE_X(x) = \mu + \frac{h}{\lambda r'(\theta)}, \tag{3.6}$$

where

$$h = \frac{\partial}{\partial\theta}\log S(x \mid \theta, \lambda)$$

Proof. Recall that the tail function is

$$S(x \mid \theta, \lambda) = \int_x^\infty e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} x dQ_\lambda^{**}(x),$$

and

$$h = \frac{\partial}{\partial\theta}\log S(x \mid \theta, \lambda) = \lambda r'(\theta) TCE_X(x) - \frac{\partial\kappa(\lambda,\theta)}{\partial\theta}$$

with $dP_{\theta,\lambda}^{**} = e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} dQ_\lambda^{**}(x)$. This formula can be rewritten as

$$TCE_X(x) = \mu + \frac{h}{\lambda r'(\theta)}.$$

IV. CONCLUSION

This paper examines tail conditional expectations for loss random variables that belong to the class of generalized exponential dispersion models. For the exponential dispersion models in [12], it has both the reproductive form and additive form. We extended the exponential dispersion families as GEDF based on these two forms. The representations for its mean and variance are derived. The tail conditional expectation is characterized. By comparing our results with those in [12], we see they share great similarity, but our new model covers wider scenarios.

we considered the two generalizations simultaneously to derive a more general and complicated distribution. In this case, the general form of the distribution is

$$dP_{\theta,\lambda}^{**} = e^{\lambda r(\theta)x - \kappa(\lambda,\theta)} dQ_\lambda^{**}(x).$$

We obtain $\mu = \mu(\theta) = \frac{1}{\lambda r'(\theta)} \frac{\partial\kappa(\lambda,\theta)}{\partial\theta}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2 r'(\theta)}[\frac{1}{r'(\theta)} \frac{\partial\kappa(\lambda,\theta)}{\partial\theta}]'$,

and $TCE_X(x) = \mu + \frac{h}{\lambda r'(\theta)}$.

Finally, we remark that the EDF contains a couple of special distributions that are commonly seen in probability theory and used in loss modeling. These special distributions are clearly also examples of generalized exponential dispersion models. Although it seems that no commonly seen special distributions belong to GEDF but not to EDF, it is not difficult to theoretically construct such distributions. The generalization allows more freedom to model loss variables, which may be beneficial when the losses cannot be well modeled using commonly seen special distributions.

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