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OSCILLATORY BEHAVIOR FOR A COUPLED STUART-LANDAU OSCILLATOR MODEL WITH DELAYS

Chunhua Feng

Department of Mathematics and Computer Science, Alabama State University,
Montgomery, AL, 36104, USA

Abstract: *In this paper, the oscillatory behavior of the solutions for a coupled Stuart-Landau oscillator model with delays is investigated. Time delay induced partial death patterns with conjugate coupling in relay oscillators has been investigated in the literature which is very special case because this model includes only one delay. According to the practical problem, the propagation delays are not the same as one. A model includes six different time delays is considered. By mathematical analysis method, the oscillatory behavior of the Stuart-Landau oscillators is brought to the instability of a unique equilibrium point of the model and the boundedness of the solutions. Some sufficient conditions to guarantee the existence of oscillatory solutions which are very easy to check comparing to the bifurcating method are provided. Computer simulations are given to support the present results. Our simulation suggests that time delays affect the oscillatory frequency much and amplitude slightly.*

Key words: Coupled Stuart-Landau oscillator, delay, instability, oscillation

INTRODUCTION

It is well known that the coupled dynamical systems with time-delays arise in various applications including semiconductor lasers [1-3], electronic circuits [4], optoelectronic oscillators [5], mechanical system [6, 7], neuronal networks [8-13], socioeconomic systems [14], and many others [15-24]. Recently, Sharma has investigated the following delay-coupled Stuart-Landau oscillators [25]:

$$\begin{cases} x'_{1,3} = p_{1,3}x_{1,3} - \omega y_{1,3} + \varepsilon[y_2(t-\tau_y) - x_{1,3}], \\ y'_{1,3} = p_{1,3}y_{1,3} + \omega x_{1,3} + \alpha\varepsilon[x_2(t-\tau_x) - y_{1,3}], \\ x'_2 = p_2x_2 - \omega y_2 + \varepsilon[y_1(t-\tau_y) - x_2] + \varepsilon[y_3(t-\tau_y) - x_2], \\ y'_2 = p_2 + \omega x_2 + \alpha\varepsilon[x_1(t-\tau_x) - y_2] + \alpha\varepsilon[x_3(t-\tau_x) - y_2]. \end{cases} \tag{1}$$

where $p_i = 1 - x_i^2 - y_i^2$ $(i = 1,2,3)$, ω is the intrinsic frequency of each oscillator. The parameter ε controls the conjugate coupling strength, τ_x and τ_y are the propagation delays associated with the x and y variables of the system. Initially for simplicity, the author took $\tau_x = \tau_y = \tau$, and the parameter α $(0 < \alpha < 1)$ has the potential to control and can be easily implemented in practical situations. Time delay induced partial death patterns with conjugate coupling in relay oscillators has been considered. However, $\tau_x = \tau_y = \tau$ is a very special case. In this paper we consider the following general coupled time delay model:

$$\begin{cases} x_1' = p_1x_1 - \omega_{11}y_1 + \varepsilon_{12}[y_2(t-\tau_2) - x_1], \\ y_1' = p_1y_1 + \omega_{11}x_1 + \alpha\varepsilon_{21}[x_2(t-\theta_2) - y_1], \\ x_2' = p_2x_2 - \omega_{22}y_2 + \varepsilon_{21}[y_1(t-\tau_1) - x_2] + \varepsilon_{23}[y_3(t-\tau_3) - x_2], \\ y_2' = p_2 + \omega_{22}x_2 + \alpha\varepsilon_{12}[x_1(t-\theta_1) - y_2] + \alpha\varepsilon_{32}[x_3(t-\theta_3) - y_2], \\ x_3' = p_3x_3 - \omega_{33}y_3 + \varepsilon_{32}[y_2(t-\tau_2) - x_3], \\ y_3' = p_3x_3 + \omega_{33}y_3 + \alpha\varepsilon_{32}[x_2(t-\theta_2) - y_3]. \end{cases} \tag{2}$$

where $p_i = 1 - x_i^2 - y_i^2$, time delays $\tau_i > 0$, $\theta_i > 0$, parameters $\omega_{ii} \in R$ $(i = 1,2,3)$, and $0 < \alpha < 1$. Our goal is to investigate the oscillatory behavior of the solutions for model (2). Noting that there are six different time delay values, bifurcation method is hard to deal with system (2). By means of mathematical analysis method, the dynamical behavior of system (2) has been discussed.

PRELIMINARIES

The system (2) can be expressed in the following matrix form:

$$u'(t) = Au(t) + Bu(t-\tau) + g(u(t)) \tag{3}$$

where $u(t) = (x_1(t), y_1(t), \cdots, (x_3\ t), y_3(t))^T$, $u(t-\tau) = (x_1(t-\theta_1), y_1(t-\tau_1),$ $\cdots, x_3(t-\theta_3), y_3(t-\tau_3))^T$, A and B boh are six by six matrices, and $g(u(t))$ is a six by one vector:

$$A = (a_{ij})_{6\times 6} = \begin{pmatrix} a_{11} & -\omega_{11} & 0 & 0 & 0 & 0 \\ \omega_{11} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & -\omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{22} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & -\omega_{33} \\ 0 & 0 & 0 & 0 & \omega_{33} & a_{66} \end{pmatrix},$$

where $a_{11} = 1 - \varepsilon_{12}$, $a_{22} = 1 - \alpha\varepsilon_{21}$, $a_{33} = 1 - \varepsilon_{21} - \varepsilon_{23}$, $a_{44} = 1 - \alpha\varepsilon_{12} - \alpha\varepsilon_{32}$, $a_{55} = 1 - \varepsilon_{32}$, $a_{66} = 1 - \alpha\varepsilon_{23}$.

$$B = (b_{ij})_{6\times 6} = \begin{pmatrix} 0 & 0 & 0 & \varepsilon_{12} & 0 & 0 \\ 0 & 0 & \alpha\varepsilon_{21} & 0 & 0 & 0 \\ 0 & \varepsilon_{21} & 0 & 0 & 0 & \varepsilon_{23} \\ \alpha\varepsilon_{12} & 0 & 0 & 0 & \alpha\varepsilon_{32} & 0 \\ 0 & 0 & 0 & \varepsilon_{32} & 0 & 0 \\ 0 & 0 & \alpha\varepsilon_{23} & 0 & 0 & 0 \end{pmatrix},$$

$$g(u) = (-x_1^3 - x_1y_1^2, \quad -x_1^2y_1 - y_1^3, \quad \cdots, \quad -x_3^3 - x_3y_3^2, \quad -x_3^2y_3 - y_3^3)^T.$$

The linearized system of (3) is

$$u'(t) = Au(t) + Bu(t-\tau) \tag{4}$$

Lemma 1 If matrix $M\ (= A+B)$ is a nonsingular matrix for selected parameters, then there exists a unique equilibrium point for system (2) (or (3)).

Proof Assume that $u^* = (x_1^*,\ y_1^*,\ \cdots,\ x_3^*,\ y_3^*)^T$ is an equilibrium point of system (2), then we have the following algebraic equation

$$\begin{cases} p_1{}^* x_1^* - \omega_{11} y_1^* + \varepsilon_{12}(y_2^* - x_1^*) = 0, \\ p_1{}^* y_1^* + \omega_{11} x_1^* + \alpha\varepsilon_{21}(x_2^* - y_1^*) = 0, \\ p_2{}^* x_2^* - \omega_{22} y_2^* + \varepsilon_{21}(y_1^* - x_2^*) + \varepsilon_{23}(y_3^* - x_2^*) = 0, \\ p_2{}^* y_2^* + \omega_{22} x_2^* + \alpha\varepsilon_{12}(x_1^* - y_2^*) + \alpha\varepsilon_{32}(x_3^* - y_2^*) = 0, \\ p_3{}^* x_3^* - \omega_{33} y_3^* + \varepsilon_{32}(y_2^* - x_3^*) = 0, \\ p_3{}^* x_3^* + \omega_{33} y_2^* + \alpha\varepsilon_{32}(x_2^* - y_3^*) = 0. \end{cases} \tag{5}$$

where $p_i{}^* = 1 - x_i^{*2} - y_i^{*2}\ (i = 1,2,3)$. The matrix form of (5) is the following:

$$\widetilde{M}\, u^* = \mathbf{0} \tag{6}$$

where

$$\widetilde{M} = \begin{pmatrix} m_{11} & -\omega_{11} & 0 & \varepsilon_{12} & 0 & 0 \\ \omega_{11} & m_{22} & \alpha\varepsilon_{21} & 0 & 0 & 0 \\ 0 & \varepsilon_{21} & m_{33} & -\omega_{22} & 0 & 0 \\ \alpha\varepsilon_{12} & 0 & \omega_{22} & m_{44} & \alpha\varepsilon_{32} & 0 \\ 0 & 0 & 0 & \varepsilon_{32} & m_{55} & -\omega_{33} \\ 0 & 0 & \alpha\varepsilon_{23} & 0 & \omega_{33} & m_{66} \end{pmatrix},$$

$m_{11} = 1 - x_1^{*2} - y_1^{*2} - \varepsilon_{12}$, $m_{22} = 1 - x_1^{*2} - y_1^{*2} - \alpha\varepsilon_{21}$, $m_{33} = 1 - x_2^{*2} - y_2^{*2} - \varepsilon_{21} - \varepsilon_{23}$,

$m_{44} = 1 - x_2^{*2} - y_2^{*2} - \alpha\varepsilon_{12} - \alpha\varepsilon_{32}$, $m_{55} = 1 - x_3^{*2} - y_3^{*2} - \varepsilon_{32}$, $m_{66} = 1 - x_3^{*2} - y_3^{*2} - \alpha\varepsilon_{23}$.

Based on the basic algebraic knowledge, if $\widetilde{M}$ is a nonsingular matrix, then system (6) has a unique trivial solution. Namely, $u^* = (0,0,0,0,0,0)^T$. However, when $x_i^* = y_i^* = 0$, matrix $\widetilde{M}$ changes to M (=A+B). The proof is completed.

Lemma 2 All solutions of system (2) are bounded.

Proof To prove the boundedness of the solutions in system (2), we construct a Lyapunov function $V(t) = \sum_{i=1}^{3} \frac{1}{2}\left(x_i^2 + y_i^2\right)$. Calculating the derivative of $V(t)$ through system (2) we get

$$V'(t)|_{(2)} = \sum_{i=1}^{3} (x_i' x_i + y_i' y_i)$$

$$\leq \sum_{i=1}^{3} (|a_i| x_i^2 + |b_i| y_i^2) + \sum_{i\neq j} |k_{ij}| |x_i y_j| - \sum_{i=1}^{3} (x_i^4 + 2x_i^2 y_i^2 + y_i^4) \qquad (7)$$

where a_i, b_i and k_{ij} are some constants. Obviously, when $x_i \to \infty,\ y_i \to \infty\ (i = 1, 2, 3)$, $x_i^4,\ x_i^2 y_i^2,\ y_i^4$ are higher order infinity than x_i^2, y_i^2 and $x_i y_j$, respectively. Therefore, there exists suitably large $L > 0$ such that $V'(t)|_{(2)} < 0$ as $x_i > L,\ y_i > L$. This means that all solutions of system (2) are bounded.

OSCILLATION OF THE SOLUTIONS

Theorem 1 Assume that zero is the unique equilibrium point of system (2) (or(3)) for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_5, \alpha_6$ and $\beta_1, \beta_2, \cdots, \beta_5, \beta_6$ be characteristic values of matrix *A* and *B*, respectively. If the real part of α_i and β_i $(\mathrm{i} = 1,2,\cdots,6)$ are negative, then the trivial solution is stable. If there exists some $Re(\alpha_k) > 0$ with $Re(\alpha_k) > Re(|\beta_k|)$ or $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, then the unique equilibrium point of system (3) is unstable. System (3) generates an oscillatory solution.

Proof According to the basic time delay differential equation theory, If the real part of α_i and β_i $(i = 1,2,\cdots,6)$ are negative, then the trivial solution is stable. Obviously, if the trivial solution of system (4) is unstable, then the trivial solution of system (3) is also unstable. Therefore, we only need to prove the instability of the trivial solution of system (4). Consider an auxiliary system of (4) as follows:

$$u'(t) = Au(t) + Bu(t - \tau_*) \qquad (8)$$

where $u(t-\tau_*) = (x_1(t-\tau_*),\ y_1(t-\tau_*),\ \cdots, x_3(t-\tau_*),\ y_3(t-\tau_*))^T$, and $\tau_* = \min_{1\leq i\leq 3}\{\tau_i, \theta_i\}$. If the trivial solution of system (8) is unstable, then the trivial solution of system (4) is unstable according to the property of delayed differential equation [26]. So we only need to show the instability of the trivial solution of system (8).

Since $\alpha_1, \alpha_2, \cdots, \alpha_5, \alpha_6$ and $\beta_1, \beta_2, \cdots, \beta_5, \beta_6$ are characteristic values of matrix *A* and *B*, respectively. Then the characteristic equation corresponding to system (8) is the following:

$$\prod_{i=1}^{6} (\lambda - \alpha_i - \beta_i e^{-\lambda\tau_*}) = 0 \qquad (9)$$

So, we are led to an investigation of the nature of the roots for some $k \in \{1,2,\cdots,6\}$:

$$\lambda - \alpha_k - \beta_k e^{-\lambda\tau_*} = 0 \qquad (10)$$

System (10) is a transcendental equation which is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of equation (10) under the assumption of Theorem 1. If $Re(\alpha_k) > 0$ with $Re(\alpha_k) > Re(|\beta_k|)$ hold, let $\lambda = \sigma + i\gamma$, $\alpha_k = \alpha_{k1} + i\alpha_{k2}$, $\beta_k = \beta_{k1} + i\beta_{k2}$.

Separating the real and imaginary parts from equation (10), we have

$$\sigma = \alpha_{k1} + \beta_{k1} e^{-\sigma\tau_*} \cos(\gamma\tau_*) + \beta_{k2} e^{-\sigma\tau_*} \sin(\gamma\tau_*) \tag{11}$$

$$\gamma = \alpha_{k2} - \beta_{k1} e^{-\sigma\tau_*} \sin(\gamma\tau_*) + \beta_{k2} e^{-\sigma\tau_*} \cos(\gamma\tau_*) \tag{12}$$

We show that equation (10) has a positive real part root. Let

$$f(\sigma) = \sigma - \alpha_{k1} - \beta_{k1} e^{-\sigma\tau_*} \cos(\gamma\tau_*) - \beta_{k2} e^{-\sigma\tau_*} \sin(\gamma\tau_*) \tag{13}$$

Obviously, $f(\sigma)$ is a continuous function of σ. Noting that $\alpha_{k1} > |\beta_{k1}|$. This means that there exists

σ_1 $(0 < \sigma_1 \ll 1)$ such that $f(\sigma_1) = \sigma_1 - \alpha_{k1} - \beta_{k1} e^{-\sigma_1\tau_*} \cos(\gamma\tau_*) - \beta_{k2} e^{-\sigma_1\tau_*} \sin(\gamma\tau_*) < 0$.

Since $e^{-\sigma\tau_*} \to 0$ as $\sigma \to +\infty$, so there exists a suitably large $\sigma_2 (> 0)$ such that $f(\sigma_2) = \sigma_2 - \alpha_{k1} - \beta_{k1} e^{-\sigma_2\tau_*} \cos(\gamma\tau_*) - \beta_{k2} e^{-\sigma_2\tau_*} \sin(\gamma\tau_*) > 0$. By means of the Intermediate Value Theorem, there exists a $\bar{\sigma} \in \{\sigma_1, \ \sigma_2\}$ such that $f(\bar{\sigma}) = 0$. This means that the characteristic value λ has a positive real part. Thus, the trivial solution of system (8) is unstable, implying that the trivial solution of system (3) is unstable. Since all solutions of system (3) are bounded, the instability of the trivial solution and the boundedness of the solutions will force system (3) to generate an oscillatory solution. For the case of $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, equation (8) will have a positive root, the proof is similar and we omit it.

Theorem 2 Assume that zero is the unique equilibrium point of system (3) for selecting parameter values. Let $p = \max_{1\le i\le 6}\{a_{ii} + |\omega_{ii}|\}$, $q = \max\{|\varepsilon_{12}| + |\varepsilon_{32}|, \ |\varepsilon_{21}| + |\varepsilon_{23}|\}$. If the following inequality holds:

$$p + q > 0 \tag{14}$$

then system (3) has an oscillatory solution.

Proof We still prove that the trivial solution of system (8) is unstable. Let $v(\mathrm{t}) = \sum_{i=1}^{3}(|x_i| + |y_i|)$, from the definition of *p* and *q* we have

$$v'(t) \le pv(t) + qv(t - \tau_*) \tag{15}$$

Consider the scalar differential equation

$$z'(t) = pz(t) + qz(t - \tau_*) \tag{16}$$

According to the comparison theorem of differential equation, we have $v(t) \le z(t)$. For equation (16), the characteristic equation associated with (16) is given by

$$\lambda = p + qe^{-\lambda\tau_*} \tag{17}$$

We claim that there exists a positive characteristic root of equation (17). Indeed, let $h(\lambda) = \lambda - p - qe^{-\lambda\tau_*}$.

Then $h(\lambda)$ is a continuous function of λ. From condition (14), we have $h(0) = -p - q < 0$. On the other hand, there exists a suitably large positive λ, say λ_1 such that $h(\lambda_1) = \lambda_1 - p - qe^{-\lambda_1 \tau_*} > 0$. Again from the Intermediate Value Theorem, there exists a $\lambda^* \in \{0, \lambda_1\}$ such that $h(\lambda^*) = 0$. In other words, λ^* is a positive characteristic root of equation (17), implying that the trivial solution of equation (16) is unstable. Since $v(t) \le z(t)$, this means that the trivial solution of equation (15), thus the system (4) is unstable. It suggested that system (2) (or (3)) has an oscillatory solution.

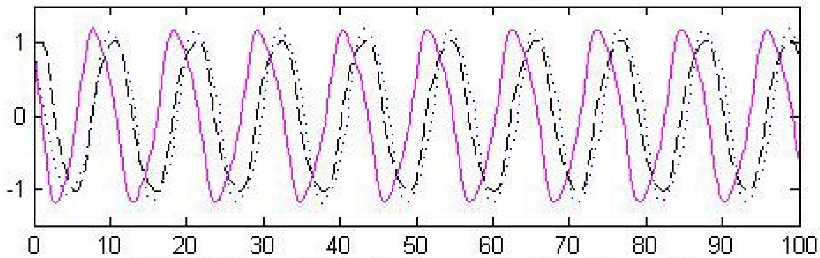
SIMULATION RESULTS

The simulation is based on the system (2), first the parameters are selected as follows: $a = 0.15$, $\omega_{11} = 0.65$, $\omega_{22} = 0.55$, $\omega_{33} = 0.75$, $\varepsilon_{12} = -0.25$, $\varepsilon_{21} = 0.35$, $\varepsilon_{23} = -0.28$, $\varepsilon_{32} = 0.32$, time delays $\theta_1 = 0.75$, $\tau_1 = 0.7$, $\theta_2 = 0.65$, $\tau_2 = 0.55$, $\theta_3 = 0.68$, $\tau_3 = 0.64$. Then the characteristic values of matrix *A* are $1.0950 \pm 0.6312\, i$, $0.9550 \pm 0.5494\, i$, $0.8350 \pm 0.7338\, i$, the characteristic values of matrix *B* are ± 0.1587, ± 0.1510, ± 0.0001. Since all characteristic value of matrix *B* are real numbers, so $\mathrm{Re}(\beta_k) = \beta_k$, and $\mathrm{Re}\,(\alpha_1) = 1.0950 > \beta_1 = 0.1510$, the conditions of Theorem 1 are satisfied. There exists an oscillatory solution for system (2) (see Fig.1). In order to see the effect of parameters ω_{ii} we set $\omega_{11} = 0.325$, $\omega_{22} = 0.275$, $\omega_{33} = 0.375$, the other parameters are the same as in figure 1, we see that the oscillation is maintained, but the oscillatory frequencies are different (see Fig. 2). Then we increase time delays as $\theta_1 = 1.25$, $\tau_1 = 1.35$, $\theta_2 = 1.15$, $\tau_2 = 1.18$, $\theta_3 = 1.28$, $\tau_3 = 1.32$. The other parameters are the same as in figure 2, we see that the oscillatory behavior is still maintained (see Fig. 3). Then we change the parameters as $a = 0.75$, $\omega_{11} = 0.95$, $\omega_{22} = 0.85$, $\omega_{33} = 0.92$; $\varepsilon_{12} = 1.15$, $\varepsilon_{21} = -1.25$, $\varepsilon_{23} = 1.28$, $\varepsilon_{32} = -1.32$. Then p=4.49, q=0.03. Therefore, $p+q = 4.52 > 0$, the conditions of Theorem 2 are satisfied. When the time delays are $\theta_1 = 0.062$, $\tau_1 = 0.068$, $\theta_2 = 0.06$, $\tau_2 = 0.065$, $\theta_3 = 0.075$, $\tau_3 = 0.058$, and $\theta_1 = 1.15$, $\tau_1 = 1.25$, $\theta_2 = 1.16$, $\tau_2 = 1.12$, $\theta_3 = 1.18$, $\tau_3 = 1.24$, respectively, there exist oscillatory solutions (see Fig. 4 and Fig. 5). We pointed out that our criterion only is a sufficient condition from our simulation.

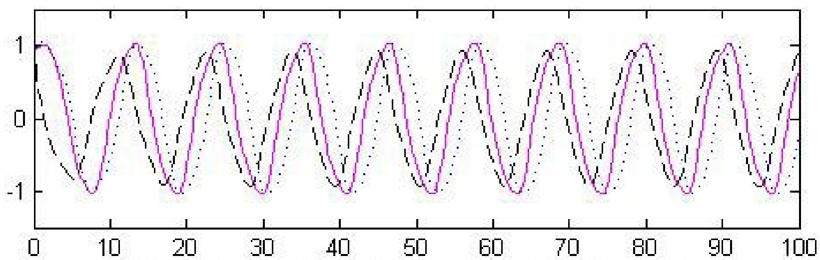
CONCLUSION

In this paper, we have discussed the oscillatory behavior of the solutions for a coupled Stuart-Landau oscillator model with delays. Based on mathematical analysis method, we provided some sufficient conditions to guarantee the existence of oscillatory solutions. Some simulations are provided to indicate the correction of the criteria.

Fig 1. Oscillation of the solutions, delays: 0.75, 0.7, 0.65, 0.55, 0.68, 0.64;
a=0.15, w_{11}=0.65, w_{22}=0.55, w_{33}=0.75.

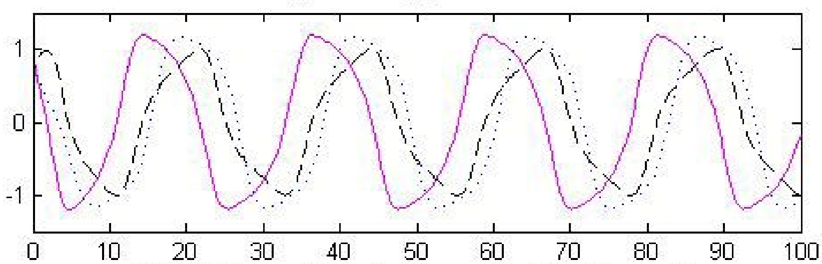


(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.

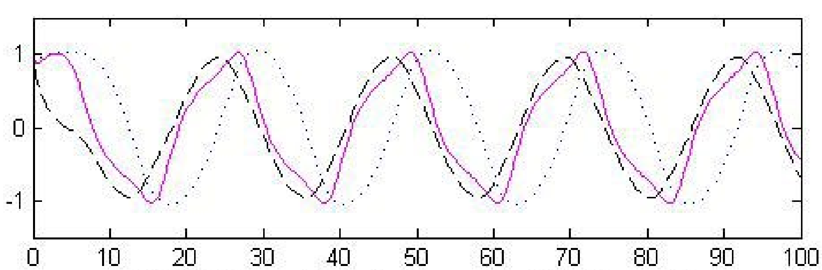


(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

Fig 2. Oscillation of the solutions, delays: 0.75, 0.7, 0.65, 0.55, 0.6, 0.64;
a=0.15, w_{11}=0.325, w_{22}=0.275, w-33=0.375.

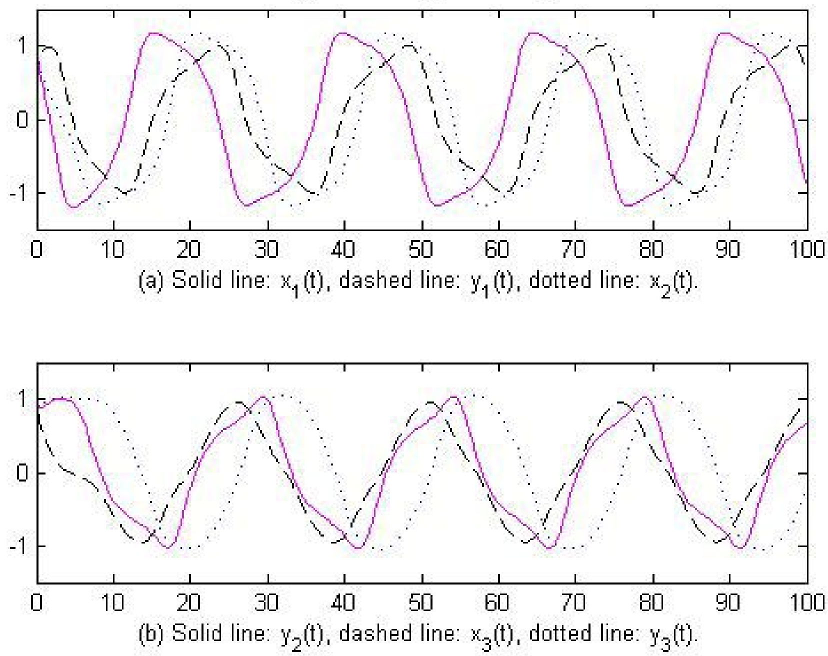


(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.



(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

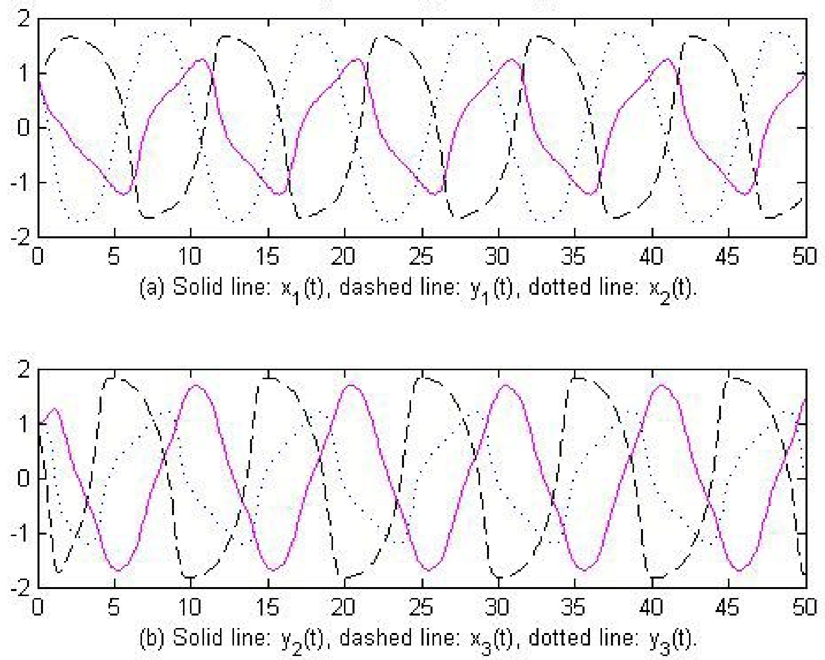
Fig 3. Oscillation of the solutions, delays: 1.25, 1.35, 1.15, 1.18, 1.28, 1.32; $a=0.15$, $w_{11}=0.325$, $w_{22}=0.275$, $w_{33}=0.375$.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.

(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

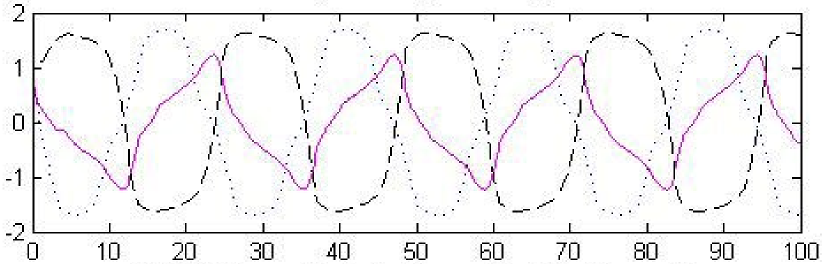
Fig 4. Oscillation of the solutions, delays: 0.062, 0.068, 0.06, 0.065, 0.075, 0.058; $a=0.75$, $w_{11}=0.95$, $w_{22}=0.85$, $w_{33}=0.92$.



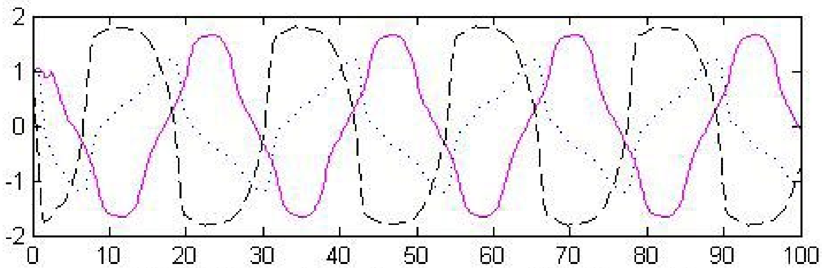
(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.

(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

Fig 5. Oscillation of the solutions, delays: 1.15, 1.25, 1.16, 1.12, 1.18, 1.24;
a=0.75, w_{11}=0.95, w_{22}=0.85, w_{33}=0.92.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.



(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

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