American Research Journal of Mathematics

Volume 7, Issue 1, 1-13 Pages
Research Article | Open Access
ISSN (Online)- 2378-704X
DOI :

Permanent Oscillation for a System of n Coupled Unbalance Damped Duffing Oscillators with Delays

Chunhua Feng

Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL, 36104, USA.

ABSTRACT

In this paper, a system of n coupled damped Duffing resonators driven by m van der Pol oscillators with delays is studied. Some sufficient conditions to ensure the permanent oscillation for the system are established. Computer simulation is given to demonstrate the result of our criterion.

KEYWORDS: *damped Duffing oscillator, van der Pol oscillator, time delay, oscillation*

INTRODUCTION

In the two decades, various isolated and coupled Duffing oscillators or Duffing-van der Pol oscillators with or without time delays have been studied extremely [1-21]. For example, Schulen et al. have investigated the following modified form of van der Pol oscillators [1]:

$$\begin{cases} \varepsilon u_i' = u_i - \frac{1}{3}u_i^3 - v_i + \frac{\lambda}{2p}\sum_{j=i-p}^{i+p}\left[b_{uu}\left(u_j(t-\tau_{ij}) - u_i\right) + b_{uv}\left(v_j(t-\tau_{ij}) - v_i\right)\right] \\ v_i' = u_i \mp a + \frac{\lambda}{2p}\sum_{j=i-p}^{i+p}\left[b_{vu}\left(u_j(t-\tau_{ij}) - u_i\right) + b_{vv}\left(v_j(t-\tau_{ij}) - v_i\right)\right] \end{cases} \tag{1}$$

where $\varepsilon = 0.05$ and $i = 1,2,\cdots,n$. The authors presented a technique to engineer solitary states which are intrigued partial synchronization patterns. The extent of displacement and the position of solitary elements can be completely controlled by the choice and positions of the incorporated delays, reshaping the delay engineered solitary states in the system. Zhang et al. have investigated three coupled van der Pol oscillators with delay as follows [2]:

$$\begin{cases} x_1'' + x_1 - \varepsilon_1(1-x_1^2)x_1' = k[x_2(t-\tau) - x_1(t-\tau)] + k[x_3(t-\tau) - x_1(t-\tau)] \\ x_2'' + x_2 - \varepsilon_1(1-x_2^2)x_2' = k[x_3(t-\tau) - x_2(t-\tau)] + k[x_1(t-\tau) - x_2(t-\tau)] \\ x_3'' + x_3 - \varepsilon_1(1-x_3^2)x_3' = k[x_2(t-\tau) - x_3(t-\tau)] + k[x_1(t-\tau) - x_3(t-\tau)] \end{cases} \tag{2}$$

By using a symmetric Hopf bifurcation theory, the Hopf bifurcations at zero point appear as the delay increases and the existence of multiple periodic solutions are also established. For a four coupled van der Pol oscillators without time delay system, the synchronization dynamics has been discussed. The stability boundaries and the main dynamical states are reported on the stability maps in the plane [3]. Wang and Chen have considered a ring of coupled van der Pol oscillators with time delay coupling which are described by the following model [4]:

$$\begin{cases} u_1''(t) - \left(\alpha - \beta u_1^2(t)\right)u_1'(t) + au_1(t) = A[u_2'(t-\tau)], \\ u_2''(t) - \left(\alpha - \beta u_2^2(t)\right)u_2'(t) + au_2(t) = A[u_3'(t-\tau)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{n-1}''(t) - \left(\alpha - \beta u_{n-1}^2(t)\right)u_{n-1}'(t) + au_{n-1}(t) = A[u_n'(t-\tau)], \\ u_n''(t) - \left(\alpha - \beta u_n^2(t)\right)u_n'(t) + au_n(t) = A[u_1'(t-\tau)]. \end{cases} \tag{3}$$

By using the method of multiple scales, the amplitude equations are obtained. Two parameters including time delay and the coupling strength are chosen as the bifurcation parameters, and then the dynamical behavior arising from the bifurcation is classified qualitatively in two-parameter plane. Kwuimy and Woafo have investigated a self-sustained electromechanical system made up of an electrical implementation of a van der Pol-Duffing type oscillator driving a macro scale mass-spring-damper linear oscillator. Using fundamental laws such as Newton and Kirchhoff laws, the parts of the system are characterized and constraints for experimental realization of a prototype are defined. Experimental simulations reveal that the proposed device exhibits periodic oscillations and complex dynamics [5]. In order to understand the emergent behavior of coupled dynamical systems and develop novel naro-electro-mechanical systems devices, Randrianandrasana et al. have considered the dynamics of a periodically driven Duffing resonator coupled to a van der Pol oscillator as follows [6,7]:

$$\begin{cases} x_1''(t) + \varepsilon\mu_1 x_1'(t) + x_1(t) + \varepsilon\alpha x_1^3(t) = \varepsilon\beta(x_2(t) - x_1(t)) + \varepsilon F\cos(\Omega\tau) \\ x_2''(t) + +\varepsilon\mu_2(x_2^2(t) - 1)x_2'(t) + x_2(t) = \varepsilon\beta(x_1(t) - x_2(t)) \end{cases} \tag{4}$$

By means of bifurcating method, Leung et al. have studied the following damped Duffing resonator driven by a van der Pol oscillator [8]:

$$\begin{cases} x_1''(t) + \varepsilon_1 x_1'(t) + \Omega_1^2 x_1(t) + k_1 x_1^3(t) = k_c\big(x_2(t) - x_1(t)\big) \\ x_2''(t) + \varepsilon_2(x_2^2(t) - 1)x_2'(t) + \Omega_2^2 x_2(t) = k_c\big(x_1(t) - x_2(t)\big) \end{cases} \tag{5}$$

By solving nonlinear algebraic equations, highly accurate bifurcation frequencies for various parameters are provided. The effects of the nonlinear damping, coupling stiffness on the angular frequency and amplitude of steady state response are studied. The obtained results were in good agreement with respect to the numerical integration solutions. Motivated by the above research work, in this paper we will deal with the following mathematical model of *n* coupled damped Duffing resonators driven by *m* coupled van der Pol oscillators

$$\begin{cases} x_1''(t) + \varepsilon_1 x_1'(t) + \Omega_1^2 x_1(t) + k_1 x_1^3(t) = \sum_{j=2}^{n+m} p_{1j}\left(x_j\left(t - \tilde{\tau}_{1j}\right) - x_1(t - \tilde{\tau}_1)\right) \\ \qquad + \sum_{j=2}^{n+m} q_{1j}\left(x_j'\left(t - \tilde{\mu}_{1j}\right) - x_1'(t - \tilde{\mu}_1)\right), \\ x_2''(t) + \varepsilon_2 x_2'(t) + \Omega_2^2 x_2(t) + k_2 x_2^3(t) = \sum_{j=1, j\neq 2}^{n+m} p_{2j}\left(x_j\left(t - \tilde{\tau}_{2j}\right) - x_2(t - \tilde{\tau}_2)\right) \\ \qquad + \sum_{j=1, j\neq 2}^{n+m} q_{2j}\left(x_j'\left(t - \tilde{\mu}_{2j}\right) - x_2'(t - \tilde{\mu}_2)\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n''(t) + \varepsilon_n x_n'(t) + \Omega_n^2 x_n(t) + k_n x_n^3(t) = \sum_{j=1, j\neq n}^{n+m} p_{nj}\left(x_j\left(t - \tilde{\tau}_{nj}\right) - x_n(t - \tilde{\tau}_n)\right) \\ \qquad + \sum_{j=1, j\neq n}^{n+m} q_{nj}\left(x_j'\left(t - \tilde{\mu}_{nj}\right) - x_n'(t - \tilde{\mu}_n)\right), \\ x_{n+1}''(t) + \varepsilon_{n+1}(x_{n+1}^2(t) - 1)x_{n+1}'(t) + \Omega_{n+1}^2 x_{n+1}(t) = \sum_{j=1, j\neq n+1}^{n+m} p_{n+1,j}\left(x_j\left(t - \tilde{\tau}_{n+1,j}\right)\right. \\ \qquad \left. - x_{n+1}(t - \tilde{\tau}_{n+1})\right) + \sum_{j=1, j\neq n+1}^{n+m} q_{n+1,j}\left(x_j'\left(t - \tilde{\mu}_{n+1,j}\right) - x_{n+1}'(t - \tilde{\mu}_{n+1})\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n+m}''(t) + \varepsilon_{n+m}(x_{n+m}^2(t) - 1)x_{n+m}'(t) + \Omega_{n+m}^2 x_{n+m}(t) = \sum_{j=1,}^{n+m-1} p_{n+m,j}\left(x_j\left(t - \tilde{\tau}_{n+m,j}\right)\right. \\ \qquad \left. - x_{n+m}(t - \tilde{\tau}_{n+m})\right) + \sum_{1}^{n+m-1} q_{n+m,j}\left(x_j'\left(t - \tilde{\mu}_{n+m,j}\right) - x_{n+m}'(t - \tilde{\mu}_{n+m})\right). \end{cases} \tag{6}$$

where $x_i = x_i(t)$represents coordinate, ε_i, Ω_i^2 and $k_i$are the damping coefficient, linear frequency and nonlinear stiffness of the Duffing resonator respectively. p_{ij}, q_{ij} are the coupling linear stiffness. By means of mathematical analysis method, some sufficient conditions to ensure the permanent oscillation of system (6) were obtained. Numerical simulation is provided to support our result. It should be emphasized that if the constants ε_i, $\Omega_i^2, k_i p_{ij}$, $q_{ij}, \tilde{\tau}_{ij}$ and $\tilde{\mu}_{ij}$ are different values, the method of Hopf bifurcation is very hard to deal with system (6). This is due to the complexity of finding the bifurcating parameter.

PRELIMINARIS

Let $\tau_1 = \tilde{\tau}_1, \tau_3 = \tilde{\tau}_2, \cdots, \tau_{2k-1} = \tilde{\tau}_k, \tau_2 = \tilde{\mu}_1, \tau_4 = \tilde{\mu}_2, \cdots, \tau_{2k} = \tilde{\mu}_k$, $a_{i,2j-1} = p_{ij}$, $b_{i,2j} = q_{ij}$. It is convenient to write (6) as an equivalent 2(*n*+*m*)-dimensional first order system:

$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + \sum_{j=2}^{n+m} a_{1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_1(t-\tau_1)] \\ \quad + \sum_{j=2}^{n+m} b_{2,2j}[u_{2j}(t-\tau_{2j}) - u_2(t-\tau_2)], \\ u_3' = u_4, \\ \cdots\cdots\cdots \\ u_{2n}' = -\varepsilon_n u_{2n} - \Omega_n^2 u_{2n-1} - k_n u_{2n-1}^3 + \sum_{j=1, j\neq n}^{n+m} a_{2n-1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) \\ \quad - u_{2n-1}(t-\tau_{2n-1})] + \sum_{j=1,j\neq n}^{n+m} b_{2n,2j}[u_{2j}(t-\tau_{2j}) - u_{2n}(t-\tau_{2n})], \\ u_{2n+1}' = u_{2n+2}, \\ \cdots\cdots\cdots \\ u_{2(n+m)-1}' = u_{2(n+m)}, \\ u_{2(n+m)}' = \varepsilon_{n+m} u_{2(n+m)} - \varepsilon_{n+m} u_{2(n+m)-1}^2 u_{2(n+m)} - \Omega_{n+m}^2 u_{2(n+m)-1} \\ \quad + \sum_{j=1}^{n+m-1} a_{2(n+m)-1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_{2(n+m)-1}(t-\tau_{2(n+m)-1})] \\ \quad + \sum_{j=1}^{n+m-1} b_{2(n+m),2j}[u_{2j}(t-\tau_{2j}) - u_{2(n+m)}(t-\tau_{2(n+m)})] \end{cases} \tag{7}$$

where $u_i = u_i(t)$ $(i = 1, 2, \cdots, 2(n+m))$. The matrix form of system (7) is as follows:

$$U'(t) = PU(t) + QU(t-\tau) + \Phi(U(t)) \tag{8}$$

Where

$U(t) = [u_1(t), u_2(t), \cdots, u_{2(n+m)}(t)]^T, U(t-\tau) = [u_1(t-\tau_1), u_2(t-\tau_2), \cdots,$ $u_{2(n+m)}(t-\tau_{2(n+m)})]^T, \Phi(U(t)) = [0, -k_1u_1^3, 0, -k_2u_2^3, \cdots, -\varepsilon_{n+m}u_{2(n+m)-1}^2 u_{2(n+m)}]^T.$

$$P = (p_{ij})_{2(n+m)\times 2(n+m)} = \begin{pmatrix} 0 & 1 & & 0 & 0 & \cdots & 0 & 0 \\ -\Omega_1^2 & -\varepsilon_1 & & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & & 0 & 0 \\ 0 & 0 & -\Omega_2^2 & -\varepsilon_2 & \vdots & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & & -\Omega_{n+m}^2 & \varepsilon_{n+m} \end{pmatrix}$$

$$Q = (q_{ij})_{2(n+m)\times 2(n+m)} = \begin{pmatrix} 0 & 0 & & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & q_{23} & q_{24} & \cdots & q_{2,2(n+m)} \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} & \vdots & q_{4,2(n+m)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ q_{2(n+m),1} & \cdots & \cdots & \cdots & \cdots & q_{2(n+m),2(n+m)} \end{pmatrix}$$

where $q_{21} = -\sum_{j=2}^{n+m} a_{1,2j-1}, q_{22} = -\sum_{j=2}^{n+m} b_{2,2j}, \cdots, q_{2(n+m),2(n+m)} = b_{2(n+m),2(n+m)}$.

Obviously, the linearized system of (7) is the following:

$$U'(t) = PU(t) + QU(t-\tau) \tag{9}$$

Definition 1 A solution of system (6) is called oscillatory if the solution is neither eventually positive nor eventually negative.

Lemma 1 Assume that system (6) has a unique equilibrium point and all solutions are bounded. If the unique equilibrium point of system (6) is unstable, then system (6) generates a limit cycle. In other words, there exists a permanent oscillatory solution of system (6).

Proof See [22] and the appendix of [23].

Lemma 2 For selected parameter values ε_i, Ω_i^2, k_i, a_{ij}, b_{ij}, if the following coefficient matrix M of system (12) is a nonsingular matrix, then system (6) (or equivalent system (7)) has a unique equilibrium point.

Proof An equilibrium point $U^* = [u_1^*,\ u_2^*,\ \cdots,\ u_{2(n+m)}^*]^T$ of system (7) is a constant solution of the following algebraic equation

$$\begin{cases} u_2^* = 0, \\ -\varepsilon_1 u_2^* - \Omega_1^2 u_1^* - k_1 u_1^{*3} + \sum_{j=2}^{n+m} a_{1,2j-1}(u_{2j-1}^* - u_1^*) + \sum_{j=2}^{n+m} b_{2,2j}(u_{2j}^* - u_2^*) = 0, \\ u_4^* = 0, \\ -\varepsilon_2 u_4^* - \Omega_2^2 u_3^* - k_2 u_3^{*3} + \sum_{j=1,j\neq 2}^{n+m} a_{2,2j-1}(u_{2j-1}^* - u_3^*) + \sum_{j=1,j\neq 2}^{n+m} b_{4,2j}(u_{2j}^* - u_4^*) = 0, \\ \cdots\cdots \\ -\varepsilon_n u_{2n}^* - \Omega_n^2 u_{2n-1}^* - k_n u_{2n-1}^{*3} + \sum_{j=1,j\neq n}^{n+m} a_{2n-1,2j-1}(u_{2j-1}^* - u_{2n-1}^*) \\ \quad + \sum_{j=1,j\neq n}^{n+m} b_{2n,2j}(u_{2j}^* - u_{2n}^*) = 0, \\ u_{2n+2}^* = 0, \\ \cdots\cdots \\ u_{2(n+m)}^* = 0, \\ \varepsilon_{n+m} u_{2(n+m)}^* - \varepsilon_{n+m} u_{2(n+m)-1}^{*2} u_{2(n+m)}^* - \Omega_{n+m}^2 u_{2(n+m)-1}^* \\ + \sum_{j=1}^{n+m-1} a_{2(n+m)-1,2j-1}(u_{2j-1}^* - u_{2(n+m)-1}^*) + \sum_{j=1}^{n+m-1} b_{2(n+m),2j}(u_{2j}^* - u_{2(n+m)}^*) = 0. \end{cases} \tag{10}$$

Since $u_2^* = 0, u_4^* = 0, \cdots, u_{2(n+m)}^* = 0$, from (10) we have

$$\begin{cases} -\Omega_1^2 u_1^* + \sum_{j=2}^{n+m} a_{1,2j-1}(u_{2j-1}^* - u_1^*) = k_1 u_1^{*3}, \\ -\Omega_2^2 u_3^* + \sum_{j=1,j\neq 2}^{n+m} a_{2,2j-1}(u_{2j-1}^* - u_3^*) = k_2 u_3^{*3}, \\ \cdots\cdots \\ -\Omega_{n+m-1}^2 u_{2(n+m)-3}^* + \sum_{j=1,j\neq n+m-1}^{n+m} a_{2(n+m)-3,2j-1}(u_{2j-1}^* - u_{2(n+m)-3}^*) = 0, \\ -\Omega_{n+m}^2 u_{2(n+m)-1}^* + \sum_{j=1}^{n+m-1} a_{2(n+m)-1,2j-1}(u_{2j-1}^* - u_{2(n+m)-1}^*) = 0. \end{cases} \tag{11}$$

We first consider the homogeneous system associated with system (11) as follows:

$$\begin{cases} -\Omega_1^2 u_1^* + \sum_{j=2}^{n+m} a_{1,2j-1}(u_{2j-1}^* - u_1^*) = 0, \\ -\Omega_2^2 u_3^* + \sum_{j=1,j\neq 2}^{n+m} a_{2,2j-1}(u_{2j-1}^* - u_3^*) = 0, \\ \cdots\cdots \\ -\Omega_{n+m-1}^2 u_{2(n+m)-3}^* + \sum_{j=1,j\neq n+m-1}^{n+m} a_{2(n+m)-3,2j-1}(u_{2j-1}^* - u_{2(n+m)-3}^*) = 0, \\ -\Omega_{n+m}^2 u_{2(n+m)-1}^* + \sum_{j=1}^{n+m-1} a_{2(n+m)-1,2j-1}(u_{2j-1}^* - u_{2(n+m)-1}^*) = 0. \end{cases} \tag{12}$$

Since M is a non-singular matrix, the determinant of the coefficient matrix of system (12) does not equal to zero. According to the algebraic basic theorem, system (12) implies that $u_1^* = 0, u_3^* = 0, \cdots, u_{2(n+m)-1}^* = 0$. In other words, system (12) has a unique trivial solution. Noting that in system (11), $h_{2j-1}(u_{2j-1}^*) = k_j(u_{2j-1}^{*3})$ $(j = 1,2,\cdots,n)$ are monotone functions, and only $h_{2j-1}(0) = 0$. This implies that $U^* = [0,\ 0,\ \cdots,\ 0]^T$ is the unique equilibrium point of the system (6). The proof is completed.

Lemma 3 Assume that $\varepsilon_i > 0$, $k_i > 0$, then all solutions of system (6) (or equivalent system (7)) are bounded.

Proof To prove the boundedness of the solutions in system (6), we construct a Lyapunov function

$$V(t) = \frac{1}{2} \sum_{i=1}^{2(n+m)} u_i^2(t)$$

Calculating the derivative of $V(t)$ through system (6), we derive that

$$
\begin{aligned}
&D^+V(t)|_{(6)} \\
&= u_1u_2 \\
&+ u_2\left(-\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + \sum_{j=2}^{n+m} a_{1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_1(t-\tau_1)]\right. \\
&\left.+ \sum_{j=2}^{n+m} b_{2,2j}[u_{2j}(t-\tau_{2j}) - u_2(t-\tau_2)]\right) + u_3u_4 + \cdots \\
&+u_{2(n+m)}\left(\varepsilon_{n+m}u_{2(n+m)} - \varepsilon_{n+m}u_{2(n+m)-1}^2 u_{2(n+m)} - \Omega_{n+m}^2 u_{2(n+m)-1}\right. \\
&+ \sum_{j=1}^{n+m-1} a_{2(n+m)-1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_{2(n+m)-1}(t-\tau_{2(n+m)-1})] \\
&\left.+ \sum_{j=1}^{n+m-1} b_{2(n+m),2j}[u_{2j}(t-\tau_{2j}) - u_{2(n+m)}(t-\tau_{2(n+m)})]\right) \\
&= u_1u_2 - \varepsilon_1 u_2^2 - \Omega_1^2 u_1u_2 \\
&+ u_2\left(\sum_{j=2}^{n+m} a_{1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_1(t-\tau_1)]\right. \\
&\left.+ \sum_{j=2}^{n+m} b_{2,2j}[u_{2j}(t-\tau_{2j}) - u_2(t-\tau_2)]\right) + u_3u_4 + \cdots \\
&\qquad -k_1u_1^3u_2 - k_2u_3^3u_4 - \varepsilon_{n+m}u_{2(n+m)-1}^2 u_{2(n+m)}^2
\end{aligned}
$$

Noting that as $u_{2j-1}u_{2j}$ tend to positive or negative infinity, $u_{2j-1}^3u_{2j}$,$u_{2j-1}^2u_{2j}^2$ are higher order positive infinity. Since $\varepsilon_i > 0,\ \ k_i > 0,$ this means that there exists suitably large $L > 0$ such that $D^+V(t)|_{(6)} < 0$ as $|u_{2j-1}| > L$ and $|u_{2j}| > L$. It suggests that all solutions of system (6) (or equivalent system (7))are bounded.

For a matrix $D = (d_{ij})_{n\times n}$, we adopt the following norm of the matrix $\|D\| = \max_{1\le j\le n}|d_{ij}|$ and the matrix measure $\mu(D) = \max_{1\le j\le n}(d_{jj} + \sum_{i=1,i\ne j}^{n}|d_{ij}|)$[24].

THE EXISTENCE OF PERMANENT OSCILLATORY SOLUTIONS

Theorem 1 Suppose that system (7) has a unique equilibrium point and all solutions are bounded for selecting parameters. Let $\alpha_1,\alpha_2,\cdots,\alpha_{2(n+m)}$and $\beta_1,\beta_2,\cdots,\beta_{2(n+m)}$denote the eigenvalues of matrices *P* and *Q*, respectively. $\alpha_i = \alpha_{i1} + i\,\alpha_{i2}$ *(*α_{i2} *may be zero)*, $\beta_i = \beta_{i1} + i\,\beta_{i2}$ *(*β_{i2} *may be zero)*, then

(1) For some k($k \in \{1,2,\cdots,2(n+m)\}$, $\alpha_{k1} > 0$or

(2) All $\alpha_{i1} < 0$, there is at least one k such that Re (β_{k1}) > |Re (α_{k1})|.

Then the trivial solution of system (9) is unstable and system (7) generates a permanent oscillatory solution.

Proof To avoid unnecessary complexity, corresponding system (9) we consider the following auxiliary system as $\tau_i = \tau_*(i = 1,2,\cdots,2(n+m))$

$$U'(t) = PU(t) + QU(t-\tau_*) \tag{14}$$

where $\tau_* \le \min\{\tau_1,\ \tau_2,\ \cdots,\tau_{2(n+m)}\}$. The characteristic equation of (14) is the following:

$$\det[\lambda E - P - Qe^{-\lambda\tau_*}] = 0 \tag{15}$$

where *E* is the $2(n+m)$ by $2(n+m)$ unit matrix. Immediately we have that

$$\prod_{j=1}^{2(n+m)}[\lambda - \alpha_j - \beta_j e^{-\lambda\tau_*}] = 0 \tag{16}$$

If we let $\lambda = \sigma + i\omega$ be the eigenvalue of system (16), then for some $\alpha_{k1} > 0$we get

$$\begin{cases}\sigma - \alpha_{k1} - \beta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) = 0\\ \omega - \alpha_{k2} - \beta_{k2}e^{-\sigma\tau_*}\sin(\omega\tau_*) = 0\end{cases} \tag{17}$$

We shall show that $\sigma > 0$ and there is an eigenvalue which has a positive real part for system (16). Indeed, let $f(\sigma) = \sigma - \alpha_{k1} - \beta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*)$, then $f(\sigma)$is a continuous function about σ.Since $\alpha_{k1} > 0$, one can select a suitable delay τ_* such that $\beta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) > -\alpha_{k1}$. Therefore, $f(0) = -\alpha_{k1} - \beta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) < 0$. On the other hand, there exists a suitably large $\bar{\sigma}\ (> 0)$ such that $f(\bar{\sigma}) = \bar{\sigma} - \alpha_{k1} - \beta_{k1}e^{-\bar{\sigma}\tau_*}\cos(\omega\tau_*)$> 0. By the continuity of $f(\sigma)$, there exists a positive $\sigma^* \in (0,\ \bar{\sigma})$ such that $f(\sigma^*) = 0$. If $\alpha_{k1} < 0$, there is a β_{k1}> $|\alpha_{k1}|$. We still have $f(0) = -\alpha_{k1} - \beta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) < 0$ holds. Thus, there is an eigenvalue of the characteristic equation associated with system (14) which has a positive real part. This means that the trivial solution of system (14) is unstable, implying that the trivial solution of system (9) for $\tau_1 = \tau_2 = \cdots = \tau_{2(n+m)} = \tau_*$ is unstable. For a time delayed system, if the trivial solution is unstable, then the instability of the trivial solution will be maintained as time delay increases. Therefore, for any delays in the system (9), thus the system (7) is also unstable. Based on Lemma 1, this implies that system (7) generates a permanent oscillatory solution. We select a suitable delay such that the system has an oscillatory solution, this oscillation is said to induce by time delay. The proof is completed.

Theorem 2 Assume that system (7) has a unique equilibrium point and all solutions are bounded for selecting parameters. If the following condition holds

$$\|Q\|\ |\mu(P)|e^2\tau_*^2 > 4e^{|\mu(P)|\tau_*} \tag{18}$$

then the trivial solution of system (9) is unstable and system (7) generates a permanent oscillatory solution.

Proof We still prove that the trivial solution of system (14) is unstable. From (14), when each $u_i(t) > 0$ we have

$$\frac{d|U(t)|}{dt} = PU(t) + QU(t - \tau_*) \tag{19}$$

when each $u_i(t) < 0$ we have

$$\frac{d|U(t)|}{dt} = -PU(t) - QU(t - \tau_*) \tag{20}$$

Therefore, we get

$$\frac{d\sum_{i=1}^{2(n+m)}|u_i(t)|}{dt} \le \mu(P)\sum_{i=1}^{2(n+m)}|u_i(t)| + \|Q\|\sum_{i=1}^{2(n+m)}|u_i(t - \tau_*)| \tag{21}$$

Let $z(t) = \sum_{i=1}^{2(n+m)}|u_i(t)|$, we consider a scalar time delay equation

$$\frac{dz(t)}{dt} \le \mu(P)z(t) + \|Q\|z(t - \tau_*) \tag{22}$$

If the unique equilibrium point of system (22) is stable, then the characteristic equation associate with (22) given by

$$\lambda = \mu(P) + \|Q\|e^{-\lambda\tau_*} \tag{23}$$

will have a real negative root say $\lambda_0$and we have from (23)

$$|\lambda_0| \ge \|Q\|e^{-\lambda_0\tau_*} - |\mu(P)| = \|Q\|e^{|\lambda_0|\tau_*} - |\mu(P)| \tag{24}$$

Using the formula $e^x \ge \frac{e^2}{4}x^2\ (x \ge 0)$one can get

$$1 \ge \frac{\|Q\|e^{|\lambda_0|\tau_*}}{|\mu(P)| + |\lambda_0|} = \frac{\tau_*\|Q\|e^{(|\mu(P)|+|\lambda_0|)\tau_*}\cdot e^{-|\mu(P)|\tau_*}}{(|\mu(P)| + |\lambda_0|)\tau_*}$$

$$\ge \frac{\|Q\|e^2\tau_*^2e^{-|\mu(P)|\tau_*}(|\mu(P)|+|\lambda_0|)}{4} > \frac{\|Q\|e^2\tau_*^2|\mu(P)|}{4e^{|\mu(P)|\tau_*}} \tag{25}$$

The last inequality contradicts the equation (18). Therefore our claim regarding the instability of the equilibrium point of system (14) is valid. Based on Lemma 1, system (7) generates a permanent oscillatory solution.

COMPUTER SIMULATION RESULT

First we consider *n*=3 and *m*=1 as the following:

$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + \sum_{j=2}^{4} a_{1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_1(t-\tau_1)] \\ \quad + \sum_{j=2}^{4} b_{2,2j}[u_{2j}(t-\tau_{2j}) - u_2(t-\tau_2)], \\ u_3' = u_4, \\ u_4' = -\varepsilon_2 u_4 - \Omega_2^2 u_3 - k_2 u_3^3 + \sum_{j=1, j\neq 2}^{4} a_{3,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_3(t-\tau_3)] \\ \quad + \sum_{j=1, j\neq 2}^{4} b_{4,2j}[u_{2j}(t-\tau_{2j}) - u_4(t-\tau_4)] \\ u_5' = u_6, \\ u_6' = -\varepsilon_3 u_6 - \Omega_3^2 u_5 - k_3 u_5^3 + \sum_{j=1, j\neq 3}^{4} a_{5,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_5(t-\tau_5)], \\ \quad + \sum_{j=1, j\neq 3}^{4} b_{6,2j}[u_{2j}(t-\tau_{2j}) - u_6(t-\tau_6)] \\ u_7' = u_8, \\ u_8' = \varepsilon_4 u_8 - \varepsilon_4 u_7^2 u_8 - \Omega_4^2 u_7 + \sum_{j=1}^{3} a_{7,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_7(t-\tau_7)] \\ \quad + \sum_{j=1}^{3} b_{8,2j}[u_{2j}(t-\tau_{2j}) - u_8(t-\tau_8)]. \end{cases} \tag{26}$$

System(26) can be written as

$$U'(t) = P_1 U(t) + Q_1 U(t-\tau) + \Phi(U(t)) \tag{27}$$

In system (26), we select ε_1=0.065, ε_2=0.075, ε_3=0.085, ε_4=0.095; Ω_1^2=0.058, Ω_2^2=0.054, Ω_3^2=0.056, Ω_4^2=0.055; k_1=0.85, k_2=0.75, k_3=0.95; a_{13}=0.0645, a_{15}=0.0564, a_{17}=0.0575, a_{31}=0.0285, a_{35}=0.0265, a_{37}=0.0225, a_{51}=0.0285, a_{53}=0.0265, a_{57}=0.0295, a_{71}=0.0288, a_{73}=0.0264, a_{75}=0.0275; b_{24}=-0.0385, b_{26}=0.025, b_{28}=0.075, b_{42}=0.018, b_{46}=0.055, b_{48}=0.05, b_{62}=-0.085, b_{64}=0.0125, b_{68}=0.015, b_{82}=0.35, b_{84}=0.75, b_{86}=0.15. The eigenvalues of P_1 are -0.0325±0.2886i,-0.0375±0.2293i,-0.0425±0.2371i,-0.0475±0.2297i and the eigenvalues of Q_1 are -0.1144±0.2459i,-0.0623,0.0740,0,0,0,0. Obviously, 0.0740>0.0475. It is easy to check that the conditions of Lemma 2, Lemma 3 and Theorem 1 hold. When time delays are selected as τ_1=0.95, τ_2=0.86, τ_3=0.80, τ_4=0.75, τ_5=0.82, τ_6=0.75, τ_7=0.84, τ_8=0.85 and τ_1=0.85, τ_2=0.86, τ_3=0.90, τ_4=0.82, τ_5=0.85, τ_6=0.88, τ_7=0.84, τ_8=0.8, respectively, system (26) generates a permanent oscillation (see Fig. 1 and Fig.2). However, when delays are increased as and the other parameters are the same as in Fig. 2, we see that the oscillatory behavior is maintained. But the oscillatory frequency is changed (see Figures 3A and 3B).

Then we consider *n*=2 and *m*=3 as the following:

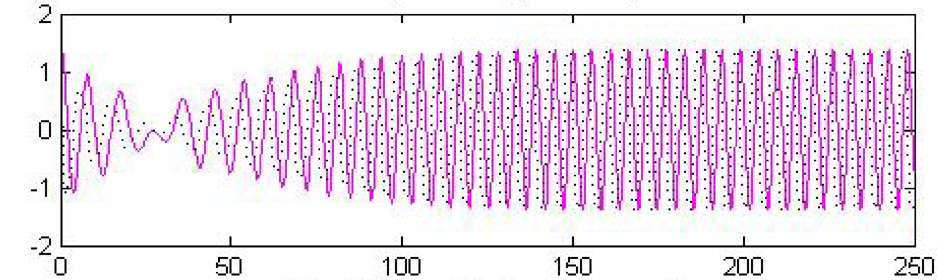
$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + \sum_{j=2}^{5} a_{1,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_1(t-\tau_1)] \\ \quad + \sum_{j=2}^{5} b_{2,2j}[u_{2j}(t-\tau_{2j}) - u_2(t-\tau_2)], \\ u_3' = u_4, \\ u_4' = -\varepsilon_2 u_4 - \Omega_2^2 u_3 - k_2 u_3^3 + \sum_{j=1, j\neq 2}^{5} a_{3,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_3(t-\tau_3)] \\ \quad + \sum_{j=1, j\neq 2}^{5} b_{4,2j}[u_{2j}(t-\tau_{2j}) - u_4(t-\tau_4)] \\ u_5' = u_6, \\ u_6' = \varepsilon_3 u_6 - \varepsilon_3 u_5^2 u_6 - \Omega_3^2 u_5 + \sum_{j=1, j\neq 3}^{5} a_{5,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_5(t-\tau_5)] \\ \quad + \sum_{j=1, j\neq 3}^{5} b_{6,2j}[u_{2j}(t-\tau_{2j}) - u_6(t-\tau_6)], \\ u_7' = u_8, \\ u_8' = \varepsilon_4 u_8 - \varepsilon_4 u_7^2 u_8 - \Omega_4^2 u_7 + \sum_{j=1, j\neq 4}^{5} a_{7,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_7(t-\tau_7)] \\ \quad + \sum_{j=1, j\neq 4}^{5} b_{8,2j}[u_{2j}(t-\tau_{2j}) - u_8(t-\tau_8)], \\ u_9' = u_{10}, \\ u_{10}' = \varepsilon_5 u_{10} - \varepsilon_5 u_9^2 u_{10} - \Omega_5^2 u_9 + \sum_{j=1}^{4} a_{9,2j-1}[u_{2j-1}(t-\tau_{2j-1}) - u_9(t-\tau_9)] \\ \quad + \sum_{j=1}^{4} b_{10,2j}[u_{2j}(t-\tau_{2j}) - u_{10}(t-\tau_{10})]. \end{cases} \tag{28}$$

System(28) can be written as

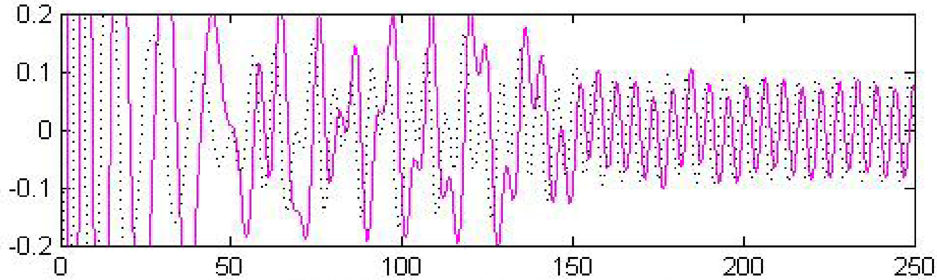
$$U'(t) = P_2 U(t) + Q_2 U(t-\tau) + \Phi(U(t)) \tag{29}$$

In system (28), when we select $\varepsilon_1=0.065$, $\varepsilon_2=0.075$, $\varepsilon_3=0.085$, $\varepsilon_4=0.095$, $\varepsilon_5=0.088$; $\Omega_1^2=0.058$, $\Omega_2^2=0.054$, $\Omega_3^2=0.056$, $\Omega_4^2=0.055$, $\Omega_5^2=0.058$; $k_1=30$, $k_2=20$; $a_{13}=-0.845$, $a_{15}=0.0564$, $a_{17}=-0.675$, $a_{19}=0.0568$, $a_{31}=-0.0785$, $a_{35}=0.0265$, $a_{37}=-0.625$, $a_{39}=0.0242$, $a_{51}=-0.0285$, $a_{53}=0.0265$, $a_{57}=-0.0295$, $a_{59}=0.0245$, $a_{71}=-0.0288$, $a_{73}=0.0264$, $a_{75}=-0.0275$, $a_{79}=0.045$, $a_{91}=-0.015$, $a_{93}=0.067$, $a_{95}=-0.025$, $a_{97}=0.045$; $b_{24}=-0.095$, $b_{26}=0.025$, $b_{28}=-0.096$, $b_{210}=0.035$, $b_{42}=-0.058$, $b_{46}=0.055$, $b_{48}=-0.072$, $b_{410}=0.045$, $b_{62}=-0.085$, $b_{64}=0.025$, $b_{68}=-0.015$, $b_{610}=0.25$, $b_{82}=-0.035$, $b_{84}=0.075$, $b_{86}=-0.045$, $b_{810}=0.045$, $b_{102}=-0.035$, $b_{104}=0.065$, $b_{106}=-0.082$, $b_{108}=0.025$. and delays are $\tau_1=2.45$, $\tau_2=2.55$, $\tau_3=3.60$, $\tau_4=3.12$, $\tau_5=2.52$, $\tau_6=2.64$, $\tau_7=2.54$, $\tau_8=2.48$, $\tau_9=2.46$, $\tau_{10}=2.42$, then $\|Q_2\|=2.012$, $|\mu(P_2)|=0.93$ and $\tau_*=2.42$. Thus, we have $\|Q_2\||\mu(P_2)|\, e^2\, \tau_*^2=2.012\times0.935\times e^2\times2.42\times2.42=81.4016$ and $4e^{|\mu(P\_2)\tau^*}=4\times e^{2.2627}=38.433$. Obviously, condition (18) holds. Based on Theorem 2, system (28) has a permanent oscillatory solution (see Fig. 4). Then we change time delays as $\tau_1=1.45$, $\tau_2=1.55$, $\tau_3=1.54$, $\tau_4=1.50$, $\tau_5=1.52$, $\tau_6=1.62$, $\tau_7=1.54$, $\tau_8=1.48$, $\tau_9=1.46$, $\tau_{10}=1.42$ and $k_1=50$, $k_2=40$, system (28) also has a permanent oscillatory solution (see Fig. 5).

Fig. 1A oscillation of the solutions, delays: 0.95, 0.86, 0.80, 0.75, 0.82, 0.75, 0.84, 0.85; k_1 = 0.85, k_2 = 0.75, k_3 = 0.95.

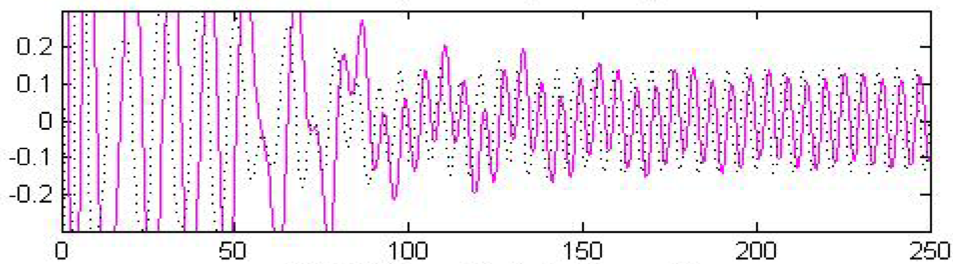


(a) Solid line: $u_1(t)$, dotted line: $u_2(t)$.

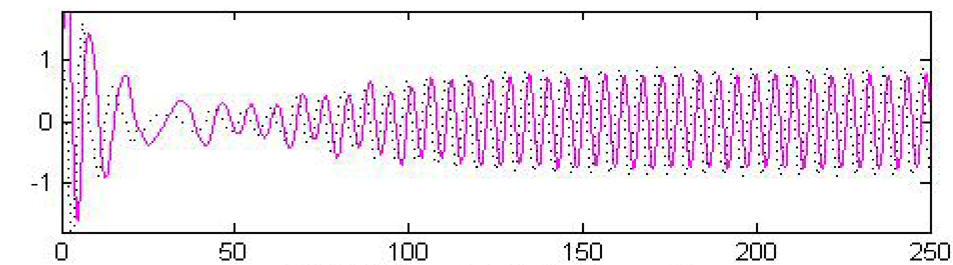


(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

Fig. 1B Oscillation of the solutions, delays: 0.95, 0.86, 0.80, 0.75, 0.82, 0.75, 0.84, 0.85; k_1= 0. 85, k_2 = 0.75, k_3 = 0.95.



(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.



(d) Solid line: $u_7(t)$, dotted line: $u_8(t)$.

Fig. 2A Oscillation of the solutions, delays: 0.85, 0.86, 0.90, 0.82, 0.85, 0.88, 0.84, 0.85; $k_1 = 0.0285$, $k_2 = 0.0275$, $k_3 = 0.0295$.

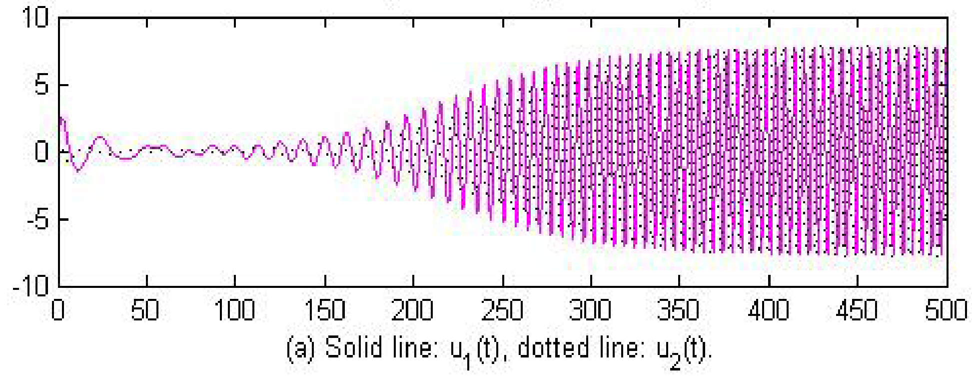


Fig. 2B Oscillation of the solutions, delays: 0.85, 0.86, 0.90, 0.82, 0.85, 0.88, 0.84, 0.85; $k_1 = 0.0285$, $k_2 = 0.0275$, $k_3 = 0.0295$.

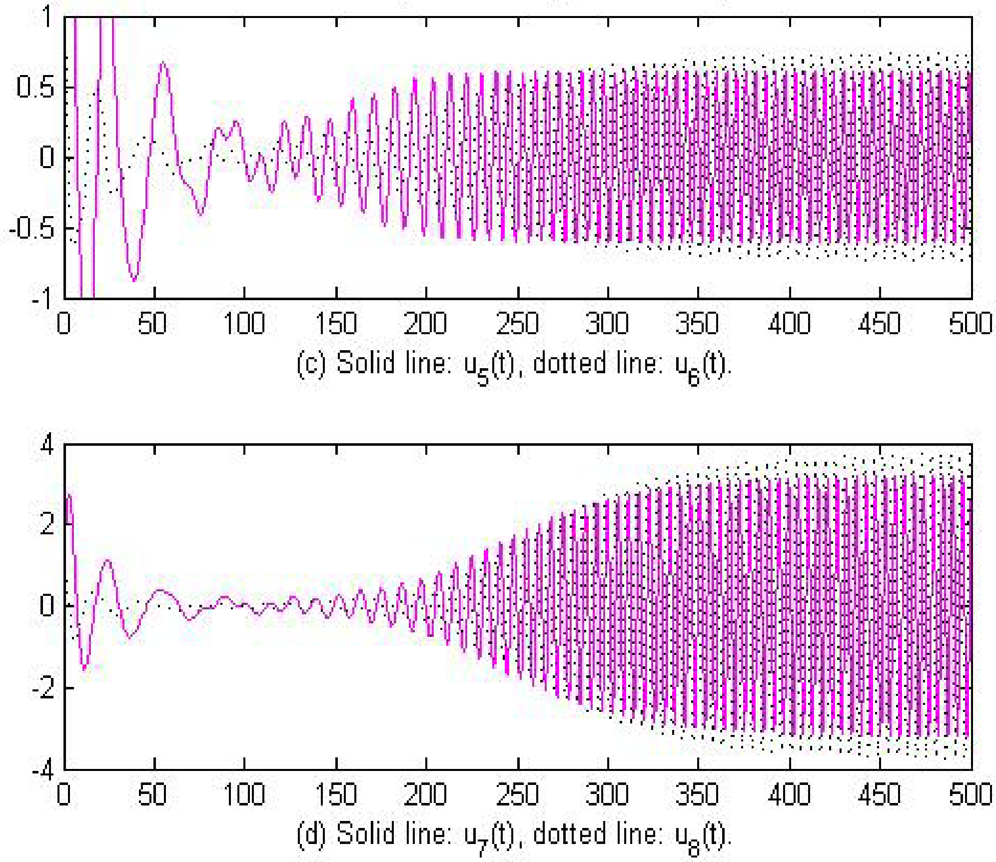
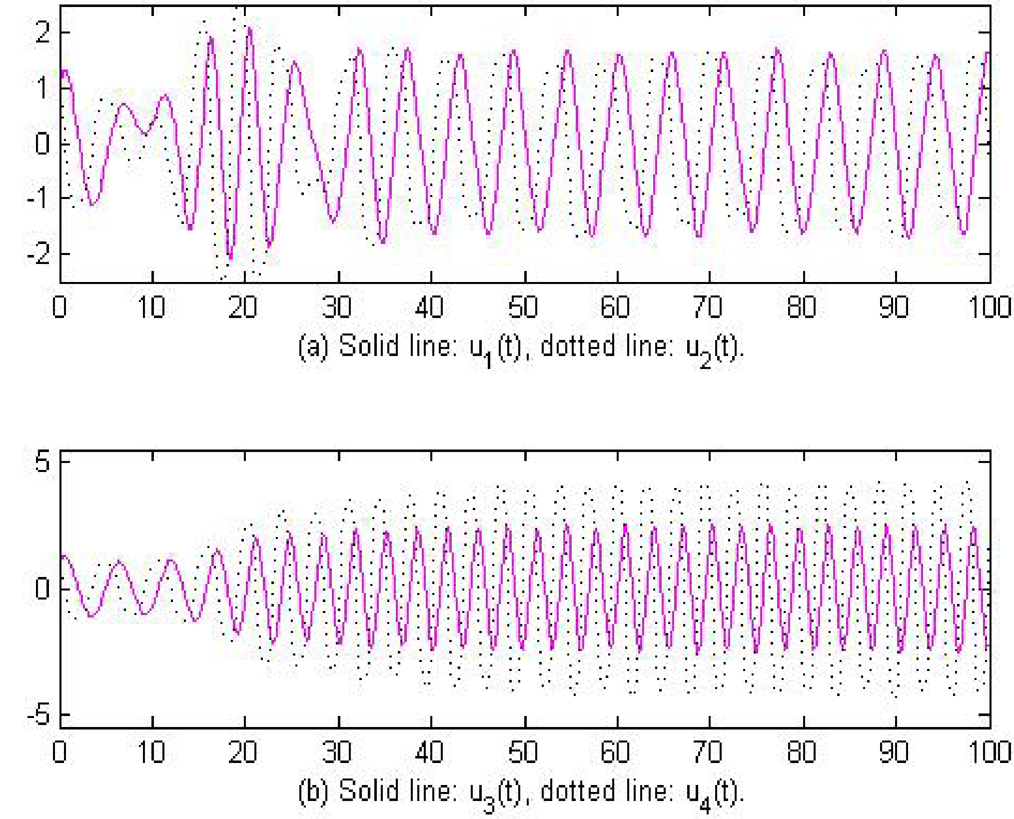


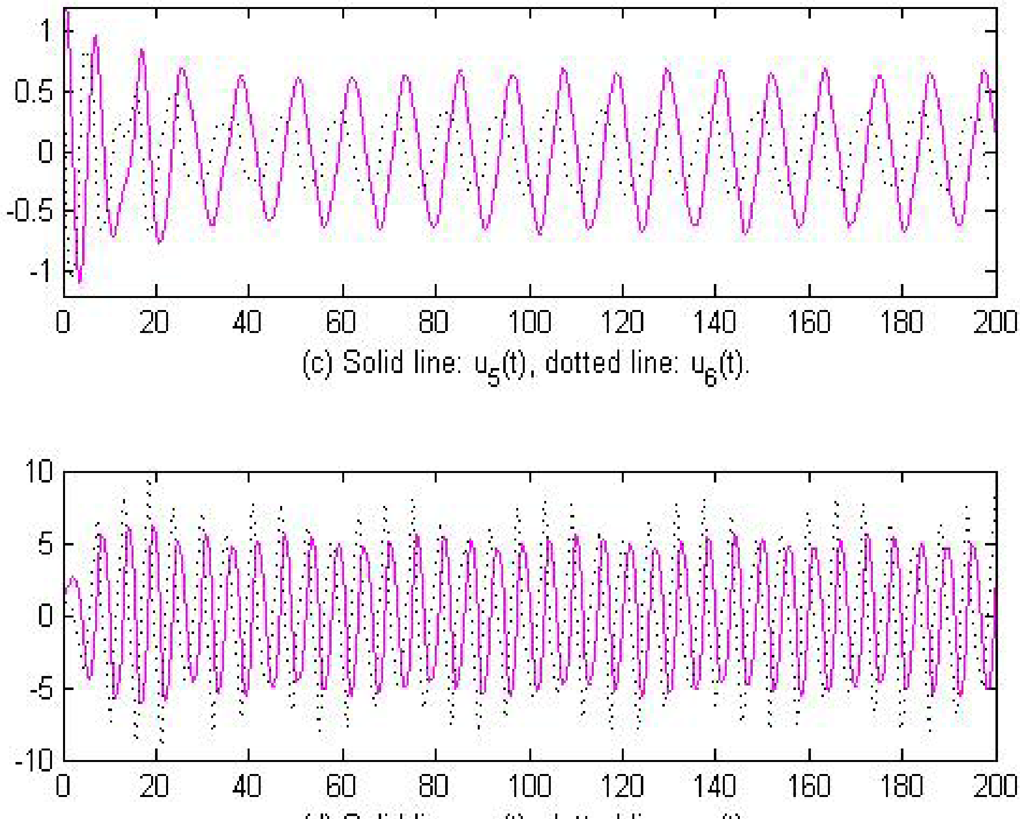
Fig. 3A Oscillation of the solutions, delays: 2.85, 1.76, 1.80, 1.78, 1.72, 1.95, 1.74, 1.85; $k_1 = 0.85$, $k_2 = 0.96$, $k_3 = 0.95$.



(a) Solid line: $u_1(t)$, dotted line: $u_2(t)$.

(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

Fig. 3B Oscillation of the solutions, delays: 2.85, 1.76, 1.80, 1.78, 1.72, 1.95, 1.74, 1.85; $k_1 = 0.85$, $k_2 = 0.96$, $k_3 = 0.95$.



(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.

(d) Solid line: $u_7(t)$, dotted line: $u_8(t)$.

Fig. 4A Oscillation of the solutions, $\tau_* = 2.42$, $k_1 = 30$, $k_2 = 20$.

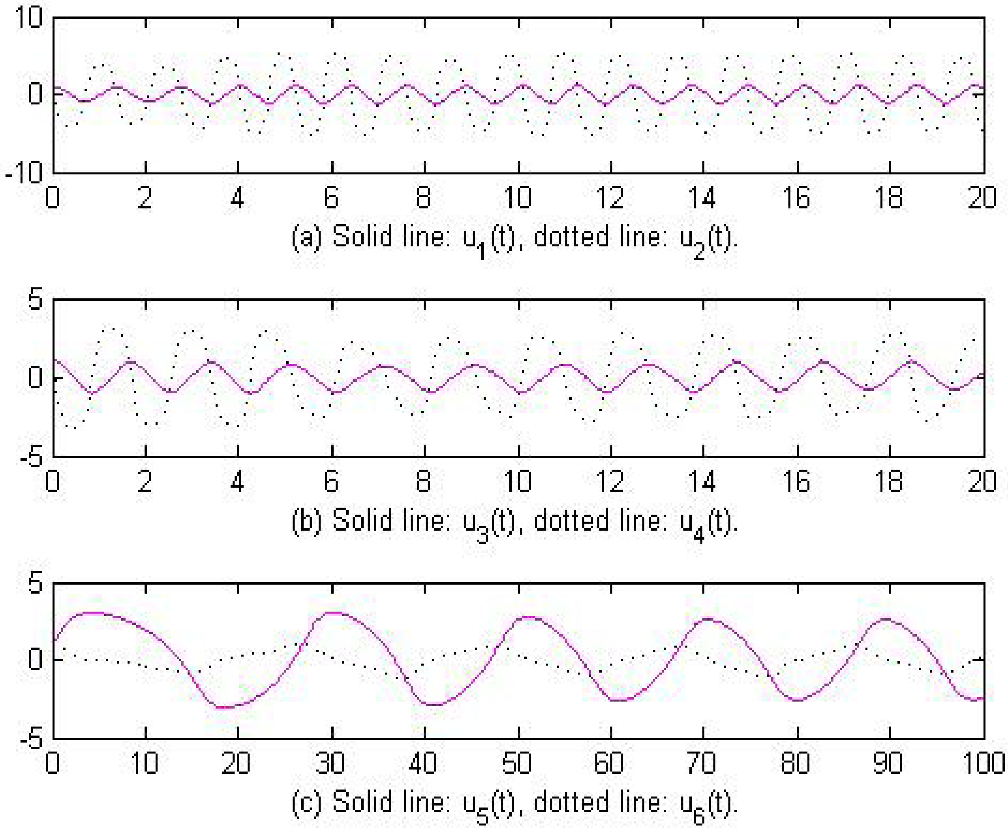


Fig. 4B Oscillation of the solutions, $\tau_* = 2.42$, $k_1 = 30$, $k_2 = 20$.

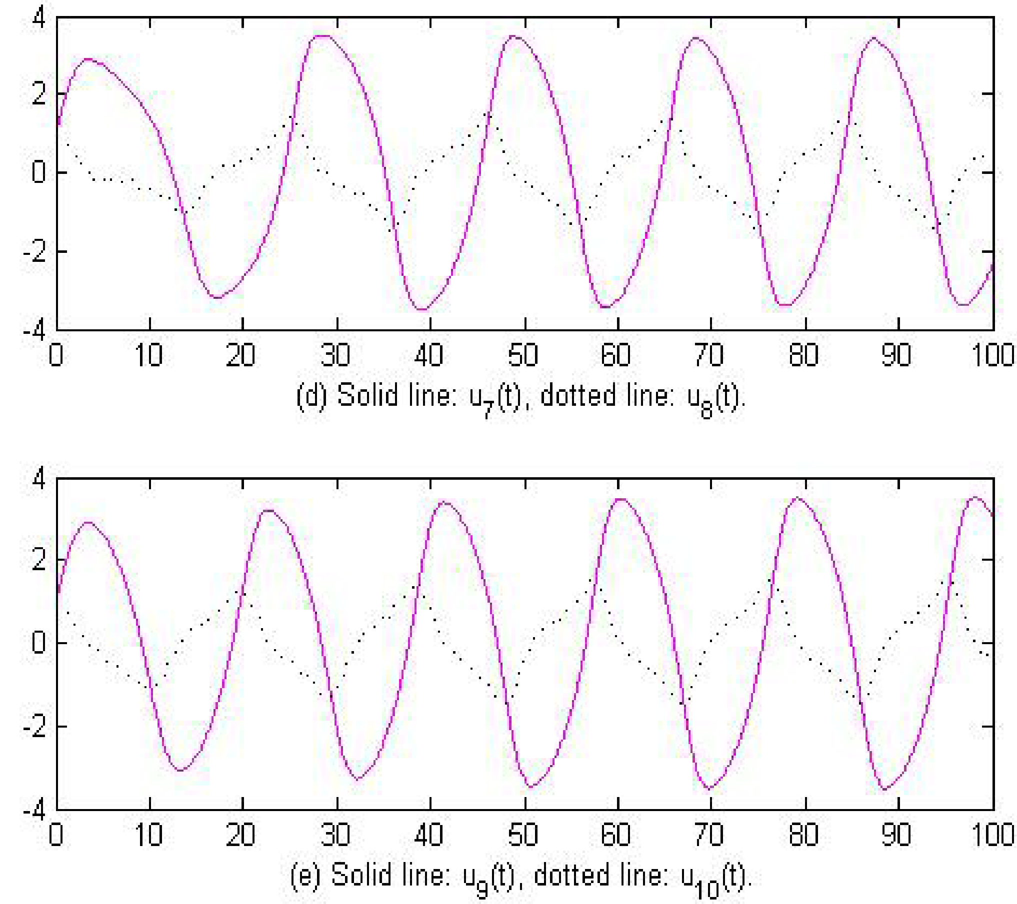


Fig. 5A Oscillation of the solutions, $\tau_* = 1.42$, $k_1 = 50$, $k_2 = 40$.

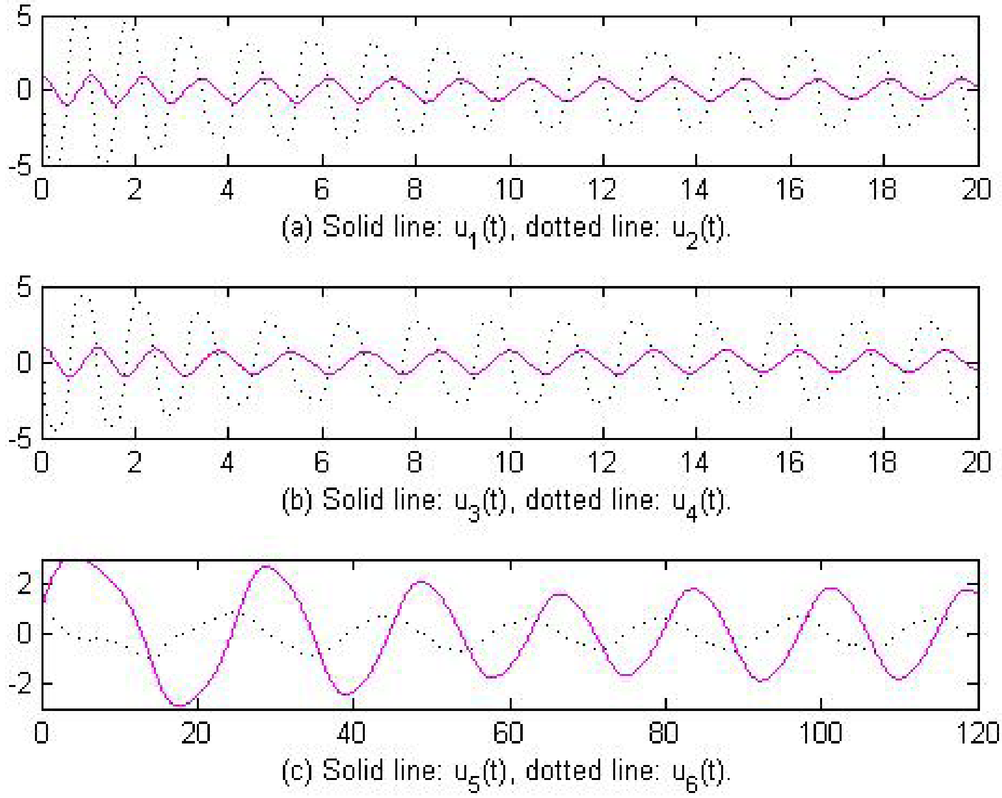
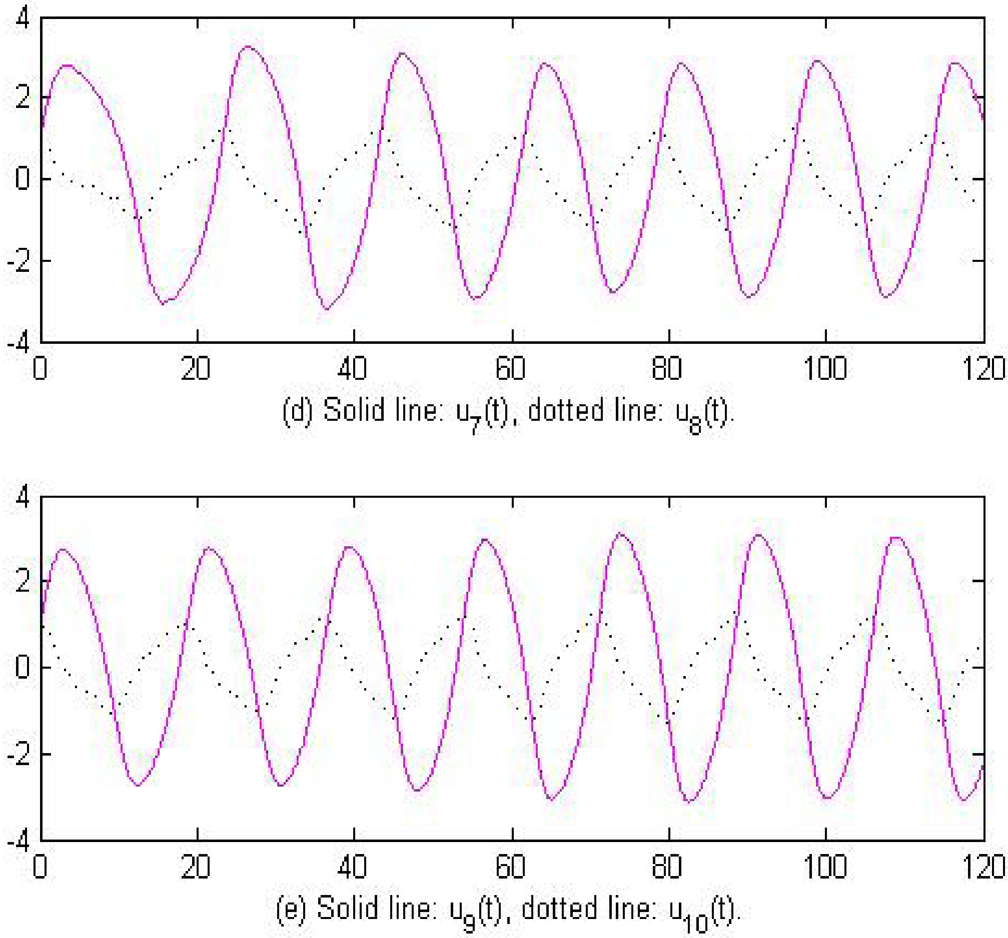


Fig. 5B Oscillation of the solutions, $\tau_* = 1.42$, $k_1 = 50$, $k_2 = 40$.



CONCLUSION

This paper discussed a system of *n* coupled damped Duffing oscillators driven by *m* van der Pol oscillators with delays. Some sufficient conditions to ensure the permanent oscillation for the system are established. This oscillation is induced by unbalance damped oscillators and time delays. When permanent oscillations occur, delays only affect the oscillation frequency. Some results in the literature have been extended.

REFERENCES

1. Schulen, L. et al. Delay engineered solitary states in complex networks. Chaos, Solitons and Fractals, 2019, 128:290-296.
2. Zhang, C. et al. Multiple Hopf bifurcations of three coupled van der Pol oscillators with delay. Appl. Math. Comput. 2011, 217:7155-7166.
3. Enjieu Kadji, H.G.et al. Synchronization dynamics in a ring of four mutuallycoupled biological systems.Commun. Nonlinear Sci. Numer. Simulat. 2008, 13:1361-1372.
4. Wang, W.Y. Chen, L.J. Weak and non-resonant double Hopf bifurcations in *m* coupled van der Pol oscillators with delay coupling. Appl. Math. Model. 2015, 39:3094-3102.
5. Kwuimy, C.A. Woafo, P. Experimental realization and simulations a self-sustained Macro Electro Mechanical System. Mechanics Research Commun. 2010, 37:106-110.
6. Randrianandrasana, M. F. et al. A preliminary study into emergent behavior in a lattice of interacting nonlinear resonators and oscillators. Commun. Nonlinear Sci. Numer. Simulat.2011, 16:2945-2956.
7. Wei, X. et al. Nonlinear dynamics of a periodically driven duffing resonator coupled to a van der Pol oscillator. Math. Prob. Engin. 2011, 10, 248328.
8. Leung, A.Y. et al. Residue harmonic balance analysis for the damped Duffing resonator driven by a van der Pol oscillator. Inter. J. Mech. Sci. 2012, 63:59-65.
9. Ghaleb, A.F. et al. Analytic approximate solutions of the cubic-quintic Duffing-van der Pol equation with two-external periodic forcing terms: Stability analysis. Math. Comput. Simul. 2021, 180:129-151.
10. Srinil, N. Zanganeh, H. Modelling of coupled cross-flow/in-line vortex-induced vibrations using double Duffing and van der Pol oscillators. Ocean Engin. 2012, 53:83-97.
11. Candido, M.R. et al. Non-existence, existence, and uniqueness of limit cycles for a generalization of the Van der Pol-Duffing and the Rayleigh-Duffing oscillators. Physica D: Nonlinear Phenom, 2020, 407, 132458.
12. Nguyen, V.D. et al. The effect of inertial mass and excitation frequency on a Duffing vibro-impact drifting system. Inter. J. Mach. Sci. 2017, 124-125:9-21.
13. Lai, Z.H. Leung, Y.G. Weak-signal detection based on the stochastic resonance of bistable Duffing oscillator and its application in incipient fault diagnosis. Mech. Syst. Signal Proc. 2016, 81:60-74.
14. Braz, A. et al. A generalization of the *S*-function method applied to a Duffing-Van der Pol forced oscillator. Comput. Physics Communic. 2020, 254, 107306.
15. Ghouli, Z. et al. Quasiperiodic energy harvesting in a forced and delayed Duffing harvester device. J. Sound Vibrat. 2017, 407:271-285.
16. Fiedler, B. et al. Coexistence of infinitely many large, stable, rapidly oscillating periodic solutions in time-delayed Duffing oscillators. J. Diff. Eqs. 2020, 268:5969-5995.
17. Shaw, P.K. et al. Antiperiodic oscillations in a forced Duffing oscillator. Chaos, Solit. Fract.2015, 78:256-266.
18. Kyziol, J. Okninski, A. Van der Pol-Duffing oscillator: Global view of metamorphoses of the amplitude profiles. Int. J. Non-Linear Mech. 2019, 116:101-106.
19. Xu, C.J. Wu, Y.S. Bifurcation control for a Duffing oscillator with delayed velocity feedback. Intern. J. Auto. Comput. 2016, 13:596-606.
20. Tchakui, M.V. Woafo, P. Dynamics of three unidirectionally coupled autonomous Duffing oscillators and application to inchworm piezoelectric motors: Effects of the coupling coefficient and delay. Chaos, 2016, 26, 113108.
21. Stachowiak, T. Hypergeometric first integrals of the Duffing and van der Pol oscillators J. Diff. Eqs. 2019, 266:5895-5911.
22. Chafee, N. A bifurcation problem for a functional differential equation of finitely retarded type. J. Math. Anal. Appl. 1971, 35:312-348.
23. Feng, C.F. Plamondon, R. An oscillatory criterion for a time delayed neural ring network model. Neural Networks, 2012,29-30:70-79.
24. Gopalsamy, K. Stability and oscillations in delay differential equations of population dynamics. Kluwer Academic Publishers, 1992.

Citation: Chunhua Feng, "Permanent Oscillation for a System of n Coupled Unbalance Damped Duffing Oscillators with Delays", American Research Journal of Mathematics, Vol 7, no. 1, 2021, pp. 1-13.