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A note on commensurability induced by oblateness at L_3

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Abstract

It is known that there exists a near one-to-one commensurability ratio between the planar angular frequencies ($s_{1,2,3}$) and the corresponding angular frequency (s_z) in the z-direction at the three collinear points (L1,2,3) in the three-dimensional restricted three-body problem. It is significant for small and practically important values of the mass parameter (μ). In this note we have used a new mean motion expression which includes the secular effects of oblateness on argument of perigee, right ascension of ascending node and mean anomaly[1], which was used in the studies[2,3,4,5]. We establish that there is ono-to-one commensurability at the external points L3 (to the right of the more massive primary), when the more massive primary is considered as an oblate spheroid with its equatorial plane coincident with the plane of motion. This study will be useful in generating the halo orbits at L3. For Saturn-Titan system, the values of the

μ and oblateness coefficient (A1) are 0.000236695 and 0.000039653936, respectively [1]. It is interesting to note that the value of A1 obtained for one-to-one commensurability is 0.0000344978 for this system, which is very close to the actual value. Thus, halo orbits of small size can be generated in Saturn-Titan system at L3.

Keywords: *Circular Restricted Three-Body Problem (CRTBP), Oblateness, Mean Motion, Lagrangian Points, One-to-one commensurability.*

Introduction

The locations of the Lagrangian points in the restricted three-body problem (CRTBP) by assuming both the primaries as oblate spheroids with their equatorial planes coincident with the plane of motion was calculated in [6]. In [7] the location of the collinear points in the same problem was studied numerically for some systems of astronomical interest. These equilibria were shown to be unstable in general, though the existence of conditional infinitesimal (linearized) periodic orbits around them was established. However, the secular effect of oblateness of the primaries on the motion of the primaries was not included. Later the oblateness of only the more massive primary was considered and the secular effect of oblateness [8] on the mean motion of the primaries was included in [9, 10, 11]. In [9, 10], the critical mass value μc was found to decrease with oblateness. In [10, 11], a numerical investigation of the locations of the five equilibrium points was made for some systems of astronomical interest. Periodic solutions of the linearized equations of motion around the five equilibrium points were studied. The angular frequency in the z-direction (s_z) was found to be more than the mean motion n. In [12] it was established that the oblateness induces a one-to-one commensurability at the exterior point L_3 and at the interior point L_2 for $0 \le \mu \le ½$ and at L_1 no such commensurability exits. L_2Series expansions were found for the long-periodic (s_4) short-periodic (s_5) orbits. s_4 was found to increase and s_5 was found to decrease with oblateness.

In this paper we have included the secular effect of oblateness on the mean anomaly, argument of perigee and right ascension of ascending node [1]. We have utilized the new mean motion to study the locations of the collinear equilibrium points. We have proved the existence of one-to-one commensurability ratio between the planar angular frequencies (s_3) and the corresponding angular frequency (s_z) in the z-direction at the collinear point (L_3).

Equations of Motion

The problem is defined in the non-dimensional pulsating synodic coordinate frame as given by Figure 1. The barycentre of the primaries mark the origin of the system which rotates about the z-axis (perpendicular to the plane of motion of primaries. The mass ratio is the ratio of the mass of less massive primary m_2 to the sum of the masses of the primaries $m_1 + m_2$) which is unity in the non-dimensional system. Point represents the point mass (with infinitesimal mass).

The equations of motion in terms of the dimensionlessframe is given by Equations (1)[11, 13]. The forcefunction Ω [9, 11] in the equations of motion is given by Equation (2). The oblateness of the more massive primary $A_1= (AE^2-AP^2)/5R^2$, AE and AP are equatorial and polar radii, respectively, and R is the distance between them, affects the force function of the system.

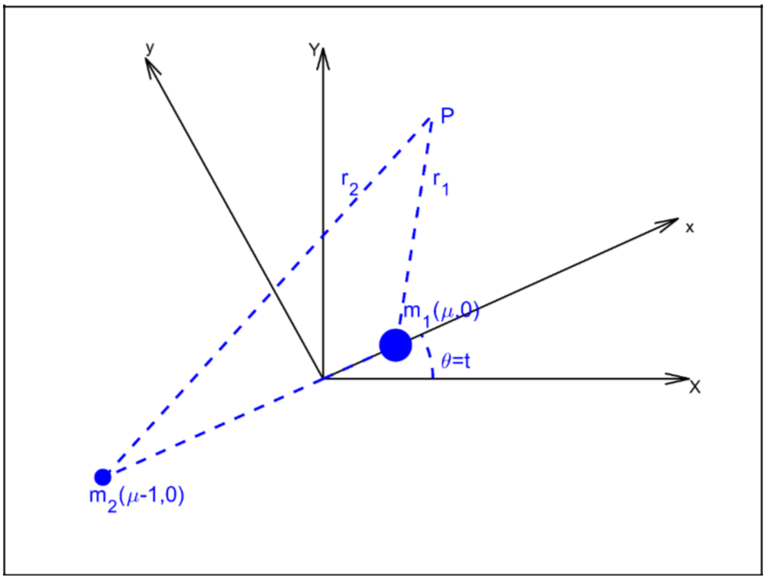

Figure 1: *Planar Restricted Three Body Problem in Dimensionless Synodic Coordinate Frame*

$$\begin{aligned} x'' - 2ny' &= \Omega_x, \\ y'' + 2nx' &= \Omega_y, \end{aligned} \tag{1}$$

$$\Omega = \frac{n^2}{2}\left((1-\mu)r_1^2 + \mu r_2^2\right) + \left[\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)A_1}{2r_1^3}\right]. \tag{2}$$

Referring Figure 1, the distances r_1 and r_2 of P from the more massive and the smaller primaries are related and to the distances x and y from the origin by

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2, \\ r_2^2 &= (x+1-\mu)^2 + y^2. \end{aligned} \tag{3}$$

Mean Motion

The mean motion equation (Equation 5) for this study is derived using the effect of J_2, given by Equations (4)

[14],on the three orbital elements – M_s- mean anomaly, ω_s - argument of perigee and Ω_s- right ascension of the ascending node.

$$\frac{dM_s}{dt} = n\left[1 + \frac{3J_2}{2a^2(1-e^2)^{\frac{3}{2}}}\right], \frac{d\omega_s}{dt} = n\left[\frac{3J_2}{a^2(1-e^2)^2}\right], \frac{d\Omega_s}{dt} = n\left[\frac{-3J_2}{2a^2(1-e^2)^2}\right]. \tag{4}$$

The mean motion n is the summation of the changes in M_s, ω_s and Ω_s after one revolution [1], given by

$$n = 1 + \frac{3A_1R^2}{2a^2[(1-e^2)Re]^2}\left(1 + \sqrt{1-e^2}\right).$$

When the value of eccentricity becomes zero and we get $n = 1 + \frac{3A_1R^2}{a^2Re^2}$, which upon non-dimensionalizing gives Equation (5) [1, 4].

$$n = 1 + 3A_1. \tag{5}$$

Location of Collinear Equilibrium Points

The equations of motion (1) are found to have singular solutions at five points [13] called the Lagrange points, liberation points or equilibrium points. Three of these equilibrium points (collinear equilibrium points - L_1, L_2 and L_3) lie in the line connecting the primaries and the other two (triangular equilibrium points - L_4 and L_5) form nearly equilateral triangles [11] with the primaries. These equilibrium points satisfy the conditions that the first derivatives of the force function equation equate to zero i.e., $\Omega_x = \Omega_y = 0$ [11, 13].

As the collinear equilibrium points lie on the x-axis, in addition to the conditions $\Omega_x = \Omega_y = 0$, they also satisfy $y = 0$. Therefore, by equating Ω_x and y to zero and making the corresponding substitutions from Equations (6), we get the seventh degree polynomials given by Equations (7), (8) and (9) for the locations of L_1, L_2 and L_3, respectively, which upon solving with the help of MATLAB for different values of μ and A_1 gives the locations of the collinear equilibrium[5]. It is interesting to note that all the three collinear points move towards the more massive primary with oblateness with the new mean motion [5]. Earlier in [11]with the mean motion$n^2 = L_1 + 3A_1/2$, it was noticed that only moves towards the bary center.

$$\begin{aligned} x_1 &= \mu - 1 - \rho_1, \\ x_2 &= \mu - 1 + \rho_2, \\ x_3 &= \mu + \rho_3, \\ y_1 &= y_2 = y_3 = 0. \end{aligned} \tag{6}$$

$$\begin{aligned} &(2+12A_1)\rho_1^7 + [(2+12A_1)(5-\mu)]\rho_1^6 + [(2+12A_1)(10-4\mu)]\rho_1^5 \\ &+ [(2+12A_1)(10-6\mu)-2]\rho_1^4 + [(2+12A_1)(5-4\mu)-4(1-\mu)]\rho_1^3 \\ &+ [9A_1(1-\mu)-12\mu]\rho_1^2 - 8\mu\rho_1 - 2\mu = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &(2+12A_1)\rho_2{}^7 - [(2+12A_1)(5-\mu)]\rho_2{}^6 + [(2+12A_1)(10-4\mu)]\rho_2{}^5 \\ &+ [(2+12A_1)(10-6\mu)-(2-4\mu)]\rho_2{}^4 \\ &\qquad + [(2+12A_1)(5-4\mu)-4+12\mu]\rho_2{}^3 \\ &+ [9A_1(1-\mu)-12\mu]\rho_2{}^2 + 8\mu\rho_2 - 2\mu = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &(2+12A_1)\rho_3{}^7 + [(2+12A_1)(2+\mu)]\rho_3{}^6 + [(2+12A_1)(1+2\mu)]\rho_3{}^5 \\ &+ [(2+12A_1)\mu-2]\rho_3{}^4 - 4(1-\mu)\rho_3{}^3 - [(2+3A_1)(1-\mu)]\rho_3{}^2 \\ &- 6(1-\mu)A_1\rho_3 - 3(1-\mu)A_1 = 0. \end{aligned} \tag{9}$$

Polynomial approximations using Taylor's series expansions for (9), we get

From (10) and (11), we notice that the equilibrium point L_3 moves towards the more massive primary with its oblateness.

$$\rho_3 = \left(1 - \frac{7}{12}\mu - \frac{1127}{20736}\mu^3 + \cdots\right) - A1\left(\frac{3}{2} - \frac{17}{24}\mu - \frac{301}{576}\mu^2 - \frac{72835}{10368}\mu^3 + \cdots\right), \tag{10}$$

and

$$x_3 = \mu + \rho_3. \tag{11}$$

As in [12], the angular frequency s_3 and $s_z(L_3)$ are given by:

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$$s_3 = [(n^2 - \eta - \beta)/2 + \{(n^2 - \eta - \beta)/2]^2 + \eta(3n^2 + 2\eta + 2\beta)\}^{1/2}]^{1/2}, \tag{12}$$

$$s_z(L_3) = (n^2 + \eta + 2\beta)^{1/2}, \tag{13}$$

$\eta = (\mu/r_1)[n^2 - 1/r_2^3]$,

$\beta = 3A_1(1 - \mu)/2r_1^5$.

r_1, r_2 are the distances of L_3 from the more massive and smaller primaries, respectively.

In view of (3), (5) and (11) using Taylor's series expansions, we get from (12) and (13):

$$s_3 = 1 + \frac{7}{8}\mu - \frac{581}{384}\mu^2 + \frac{62573}{9216}\mu^3 + A_1\left(\frac{3}{2} + \frac{259}{32}\mu - \frac{13333}{384}\mu^2 + 187.027895\mu^3\right) \tag{14}$$

$$s_z(L_3) = 1 + \frac{7}{16}\mu + \frac{161}{1536}\mu^2 + \frac{2891}{73728}\mu^3 + A_1\left(\frac{9}{2} + \frac{283}{64}\mu + \frac{2165}{768}\mu^2 - 8.075228\mu^3\right) \tag{15}$$

Commensurability at L_3

As can be seen from (14) and (15), at L_3 the in-plane frequencies s_3 is greater than s_z without oblateness, however, both s_3 and s_z increase with oblateness effect. The increase in s_z is more than in s_3 with oblateness. So there is a possibility for one-to-one commensurability between the two frequencies, for a suitable choice of A_1, for $0 \leq \mu \leq ½$.

The values of A_1 has been obtained using MATHEMATICA with an initial estimate:

The values of A_1 has been obtained using MATHEMATICA with an initial estimate:

$$A_1 = \frac{\frac{7}{16}\mu - \frac{2485}{1536}\mu^2 + \frac{497693}{73728}\mu^3}{3 - \frac{235}{64}\mu + \frac{28831}{768}\mu^2 - 195.103123417607\mu^3}, \tag{16}$$

obtained by setting $s_3 = s_z(L_3)$ from (14) and (15). It may be noted that the values of A_1 are small for small μ. For Saturn-Titan system, the values of μ and A_1 are 0.000236695 and 0.000039653936, respectively [1]. It is interesting to note that the value of A_1 obtained for one-to-one commensurability is 0.0000344978 for this system, which is very near to the actual 0.000039653936, respectively[1]. So the halo orbits of small size could be easily generated in this system. It is a very interesting result that the oblateness of the more massive primary can help in generating the halo orbits of small size around L_3 in the Saturn-Titan system. Table 1 provides the values of mass parameter (μ) and oblateness coefficient (A_1) for obtaining one-to-one commensurability for small values of μ. A_1 values are one order less than μ.

Table 1: *Values of A_1 when $s_3 = sz\ (L_3)$ for small values of μ.*

S. No	μ	A_1	$s_z = s_3$	A_1 (From Eq.)
1	10^{-6}	$0.1458329725 \times 10^{-6}$	1.00000109374913	$0.1458329725 \times 10^{-6}$
2	10^{-5}	$0.1458297254 \times 10^{-5}$	1.00001093741261	$0.1458297255 \times 10^{-5}$
3	10^{-4}	$0.1457972546 \times 10^{-4}$	1.00010936626018	$0.1457972546 \times 10^{-4}$
4	10^{-3}	$0.1454725453 \times 10^{-3}$	1.00109287498146	$0.1454725454 \times 10^{-3}$
5	10^{-2}	$0.1422377438 \times 10^{-2}$	1.01084950468990	$0.1422377438 \times 10^{-2}$

Conclusions

With the secular perturbations effects of oblateness on argument of perigee, right ascension of ascending node and mean anomaly on the mean motion [1], it is found that the mean motion increases further. The CRTBP with the more massive primary as an oblate spheroid with its equatorial plane coincident with the plane of motion is studied with the new mean motion. The locations of the three Lagrangian points of the CRTBP are computed. Series expansion for the location of L_3 is found. It is noticed that L_3 moves towards the more massive primary with the inclusion of its oblateness.

We establish that there is ono-to-one commensurability ratio between the planar angular frequency (s_3) and the corresponding angular frequency (s_z) in the z-direction at the collinear point L_3. This study will be useful in generating the halo orbits at L_3 of small size For Saturn-Titan system, the values of the μ and oblateness coefficient (A1) for this system are 0.000236695 and 0.000039653936, respectively. It is interesting to note that the value of A_1 obtained for one-to-one commensurability is 0.0000344978 for this system, which is very close to the actual value of 0.000039653936. Thus, halo orbits of small size can be generated in Saturn-Titan system at L_3. Most satellites in halo orbit serve scientific purposes, such as space telescopes.

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